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The Council and its Journals

The Scottish Mathematical Council has been supporting mathematics learning and teaching in Scotland since its founding in 1967.

The main objects of the Council are to foster and improve mathematical education at all levels and to encourage advancement and application of mathematics throughout Scotland. For this purpose the Council collaborates with educational, technological and industrial organisations and undertakes functions such as the following:

- The promotion and co-ordination of conferences, mathematical competitions, special courses of instruction and retraining schemes.
- The consideration of factors relating to the supply and deployment of qualified mathematicians with special reference to the supply of teachers.
- The giving of advice on curricula, syllabuses, examinations and other subjects of mathematical interest.
- The setting up of committees to consider and report on special problems and subjects.
- The collection and circulation of information and the publication of reports, articles and other material of pedagogic interest.
- The trusteeship of trusts established for purposes associated with the objects of the Council.

The Journal is the main instrument for reporting on the above activities. It is published annually and distributed to all secondary schools in Scotland and many interested bodies in the UK and abroad. Since 2017, the Council has also published a Primary Journal.

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Views expressed in the articles and in the Editorial in SMC Journal 49 are those of the authors and do not necessarily represent the official view of The Scottish Mathematical Council. The editors welcome the submission of suitable material for inclusion in subsequent issues. Articles should be sent to Chris Pritchard (chrispritchard2@aol.com).
Welcome to the 49th issue of the SMC Journal, the premier magazine for mathematics teachers in Scotland today. We are delighted that the cover is so striking this year and thank the Belgian mathematical artist, Jos Leys, who has graciously allowed us to use one of his Mandelbrot tributes without charge. We encourage you to visit his website at www.josleys.com/index.php.

Several articles in this issue make reference to ‘mastery’, in particular the use of concrete and pictorial representations as precursors to the introduction of abstract concepts. Concrete resources have been in schools for a very long time, yet previous attempts at using concrete materials to help pupils understand how written algorithms work hasn’t always had the desired result. Interestingly, allowing pupils to explore and make sense of concrete resources in their own ways. Is this something we have failed to do in the past? Have we simply confined their use to one of ‘demonstration tool’? In her discussion of concrete materials, Karen Oppo highlights the merits of a focus on place value partitioning and the expanded algorithm, alongside concrete materials. Could this be the ‘missing link’? If so, when demonstrating processes with concrete materials we must ensure that our choice of resource emphasises mathematical relationships and makes the relative values of the digits clear so as not to confuse pupils.

Paul McDonald, newly appointed to the SMC, has provided a detailed pedagogical argument for the inclusion of collaborative problem-solving in the mathematics classroom. After all, this is an era in which big firms develop their ideas and their products through working in teams. Stuart Welsh, in his article, explains why learning mathematics is difficult and what teachers can do to minimise that difficulty. Sadly, Stuart has now left these shores for those of the Mediterranean, a significant loss to Scotland, though he promises to keep in touch!

The experiences of a PGDE student training to be a mathematics teacher are described by Lynn McRobert. As a rite of passage into teaching many teachers have undertaken a PGDE. It may well be interesting to note what is the same and what is different between your own initial teacher education and the experience outlined by Lynn. In particular, she highlights how the partnership between the University and the schools she trained with helped her to develop an active, progressive and cohesive learning experience for her pupils during a challenging but rewarding year.

On the topic of maths in everyday life and maths at work, Francesca Iezzi describes an outreach project aimed at increasing pupils’ awareness of the impact of mathematics on a number of different careers. The article also describes how the project’s resources aim to overcome maths anxiety. Francesca argues that it is important for people to be open about how they have struggled with mathematics and how they overcame these struggles, an important lesson for teachers old and new. Meanwhile, Ayliean MacDonald has very bravely tackled the topic of Mental Health for Maths Teachers. She gives an amazingly honest description of the issues she herself has faced and argues for the removal of stigma concerning mental health as opposed to physical health. She gives clear distinctions between mental health, mental illness and mental wellbeing and provides sound advice for all teachers.

A very practical approach to proof in geometry is given in Dirk Brockmann-Behnsen’s article aptly called ‘Cut it, fold it, prove it!’ It is interesting to note the emphasis placed on the development of pupils’ ability to prove in this study undertaken in a German school and well worth comparing this with our own practice here in Scotland.

Adam McBride has done a brilliant job of converting his engaging presentation on number sequences into a concise article accessible to many pupils. We are also delighted to include a fascinating description by Helena Rocha and Isabel Oitavem of the mathematics that lies behind the use of barcodes.

Issue 49 concludes with a meaty article on complex numbers in which Chris Sangwin shows how they could be used to enlighten and to enhance understanding in other areas of the Advanced Higher curriculum. He traces their development from the sixteenth century on, finds examples in textbooks from as early as 1795 and provides copious illustrations of how complex numbers could be used imaginatively at Advanced Higher.
It is a pleasure to address you in my first report as Chair of the Scottish Mathematical Council, a post I took over from Dr Chris Pritchard in July 2019. Chris has been an extremely proactive Chair and SMC has truly grown under his leadership. We thank him for his excellent work and look forward to continuing to work with him as an Associate member of the Council. Indeed, I consider myself very fortunate to be supported by such a strong membership who give a considerable amount of their free time to support mathematics learning and teaching in Scotland.

The Scottish Mathematical Council has had another very good year, with successful conferences, the publication of its lively journals, association with a range of popular challenges and competitions and a good start to a project we've been asked to shape by the Scottish Parliament. In short, we believe we continue to provide the best support available to teachers of mathematics in Scotland, whether they be primary, secondary or F. E. practitioners.

Our annual conference at Stirling University attracted over 300 enthusiastic educators who gathered from classroom enquiries and practical ideas for use back in school. Topics included why some pupils find maths difficult and the use of concrete representations to support understanding. Heather ‘the Weather’ Reid gave an entertaining keynote on the role that mathematics has played throughout her varied career and highlighted some of the challenges and opportunities in promoting maths to the general public. The 2020 Stirling Conference will take place on Saturday 7th March and our keynote speaker will be Craig Barton (@mrbartonmaths). Save the date!

Also at the Stirling Conference we paid tribute to Brenda Harden, the fourth recipient of our Achievement Award – and very well-deserved it is too! Brenda was nominated by colleagues who praised her unrivalled passion and enthusiasm and, though now retired, Brenda continues to play an active role in the maths community, motivating and inspiring both pupils and teachers, old and new. Do you know a worthy recipient of the SMC Achievement Award? The criteria and details of how to nominate them can be found at: www.scottishmathematicalcouncil.org

In May, we held our first ever Northern Conference. Mathemagician, Andrew Jeffrey, presented an entertaining keynote and this was followed by a range of workshops. Although considerably smaller than the Stirling conference, ‘SMC North’ was deemed a success and the second Northern Conference will be held in November 2020. Neither conference could have happened without the support of Stephen Watters and Deirdre Murray. Regrettfully, Stephen and Deirdre’s term as full members ended in May and September respectively and we thank them for the excellent service they have provided to SMC over the years.

The Scottish Mathematical Council actively promotes mathematics as an enjoyable subject with problem solving at its core. To this end Mathematical Challenge, Enterprising Mathematics in Scotland and Mathématiques sans Frontières remain popular. Our heartfelt thanks to all those who work tirelessly behind the scenes to ensure the success of these competitions; however, I would like to pay special tribute to Bill Richardson, now in his twentieth year as Chair of the SMC Mathematical Challenge team.

We are also grateful to the Scottish Government whose financial support helps us sustain and extend our key activities, including the Deputy First Minister’s Daily Problems for Maths Week Scotland and Holiday Challenges, aimed at primary pupils. Our newest project is ostensibly on Mathematics and Parliament and is destined for publication on the Scottish Parliament’s Education Outreach webpages and our own website. The mathematics of democracy is most readily seen in the voting systems that are in place and in the statistics and the decisions made on the back of those statistics. So this is where we have turned our attention, at least is based on proportional representation (with certain adjustments), and this led us to consider multiplicative reasoning skills in the context of voting. How are fair elections designed? How fair are school voting systems, for example in Pupil Council elections? By what criteria is fairness judged? We’re writing material that will promote fairness whilst practising essential skills at a number of levels through the primary and secondary years.

The SMC is continually looking for ways to develop the support we offer. If you have any suggestions for how we might grow our current projects, or broaden our portfolio by introducing new activities, I would love to hear from you via email at carol_smc@icloud.com.
Mathematical Challenge continues. Numerically, the event went reasonably well. The uptake in both the Primary Division and in the Secondary Division was pretty steady. There is a real need for teachers to engage with the organisation at local and at national levels, so please send us your views on what should be done to increase pupil participation. We hope that the event is enjoyable for participants, their parents and their teachers!

The table on the next page gives a summary of the last three years. Secondary schools are informed of the problem download arrangement in a postcard sent each August. A similar mailing to primary schools is impracticable. The information for each round is at: www.wpr3.co.uk/MC/materials/

Please do what you can to encourage your local primary schools to take part. An archive of unpublished questions and solutions from around 2006 to 2018 is freely available at www.wpr3.co.uk/MC-archive/

I would welcome feedback so please send me your thoughts by e-mail to: wpr3145@gmail.com.

Section Organisers
1 Dr Richard Hepworth (University of Aberdeen)
1(P) Dr Helen Martin (University of Aberdeen)
2 Dr Jean Reinhard (University of St Andrews)
2(P) Dr Rachel Norman (University of Stirling)
3 Dr Carlos Zapata-Carratala (Edinburgh University)
3(P) Mr Eric Brown (Edinburgh)
4 Mr John Winter (University of Strathclyde)
4(P) Dr Chris Athorne (University of Glasgow)

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Mr Gregor Dickson (George Heriot’s School)
Professor Alastair Gillespie
Dr Anne Cockcroft
Mr Craig Stewart (Gryffe High School)

Once again, I should like to express my gratitude to the members of the National Committee who give of their time to run the competition. I must also thank schools and other employers for allowing the Committee to have time off to attend the three meetings held each year. The following reports are from the relevant Section Organisers.

Section 1, North: Secondary Division
Just over 100 students, from 19 schools, received awards in the North section of the Secondary Division. Of these, 52 had worked on the Junior problem set, 39 on the Middle problem set, and 14 on the Senior problem set. Many attended the Awards ceremony on 19th June 2019 in Fraser Noble Lecture Room 1, Aberdeen University. The talk was given by Assaf Libman, whose title was ‘Infinity’. The talk featured a careful analysis of what it means to count up to 5, as well as exploration of some more advanced aspects of counting. Following the talk, mugs and certificates were given out.

I am very grateful to Vicky Fletcher, Helen Gauld, Brenda Harden, and Julie McCulloch for their help.

Will Turner

Section 1, North: Primary Division
This year saw an increase in the number of schools participating, including five schools sending in entries for the first time. Also, the number of pupils taking part has continued to increase and many of these pupils are attempting the questions in all three rounds – well done.

Many interesting correct solutions were submitted for the question below and some lovely diagrams were used as explanations.

Early on a very hot day, a greengrocer places 20 kilograms of courgettes on display outside his shop. At that moment, the courgettes are 99% water. It turns out to be the hottest day of the year, and as a result, the courgettes dry out a bit. At the end of the day, the greengrocer has not sold a single courgette and the courgettes are only 98% water. What weight of courgettes does he have at the end of the day?

Thank you to the volunteers who help mark the papers, including the maths students at the University of Aberdeen.

Congratulations to the pupils who gained an award and well done to all participants. Many thanks to the parents and teachers who encourage and support our young mathematicians. Look out for further information in August for the next set of questions.

Helen Martin
Section 2, East and Central

This year we again split the secondary and primary competitions, with Rachel Norman from the University of Stirling running the primary competition and Jean Reinaud from St Andrews University running the secondary competition.

In the primary section we had 19 schools and 190 children enter with some very impressive marks. In the secondary section we had 322 entries from 20 schools which was made up of 222 juniors, 89 middle and 21 seniors so we need to encourage more seniors to take part.

The Award Day took place at the University of Stirling on the afternoon of Wednesday 5th June 2019 and was attended by about 75 award winners from 20 schools along with many teachers, parents, grandparents and friends. Mr Donald Smith, from the University of Stirling, gave an entertaining talk entitled “What comes next?” in which he explored sequences. The award winners were presented with their certificates and mugs and the event ended with
light refreshments. It is clear that the award winners found the competition stimulating and enjoyable. As several of them had been to more than one of these award days, so we encouraged them to keep on entering.

We thank colleagues from the Universities of St Andrews and Stirling for supporting the Challenge in various ways. We also thank the Institute for support for the postage and photocopying costs. As always, the markers from various university departments and schools have played an indispensable role, and we are most grateful to them.

We look forward to next year.

Jean Reinaud and Rachel Norman

Section 3, Lothian/Borders: Secondary Division
This year the organisers were Alexandre Martin, Heriot-Watt University (Secondary) and Eric Brown, University of Edinburgh (Primary).

The number of awards increased compared to last year. The quality of the entries was generally high, with cutoffs for silver and gold awards above 35, and a few students obtaining a perfect score. The marking and administering of marks was carried out in full by colleagues at Heriot-Watt University. The support and enthusiasm of everyone who participated is greatly appreciated.

Section 3, Lothian/Borders: Primary Division
The numbers of schools and entrants were much the same as last year. There were some tricky problems – Problem 1.2 involving percentages (“percentage of what?”) defeated a number of adults of my acquaintance. Some of the marking was done as always by a group of Primary teachers who have a strong commitment to Mathematics Challenge. Set 1 was marked by Moray House PGDE (Secondary) students, which was recognised as a useful contribution to their training programme.

The Gold and Silver prize winners gathered on Friday 31st May 2019. The James Watt Centre again provided an excellent venue on the attractive Heriot-Watt campus. There was a good turnout of prize winners, teachers, parents and friends. The guest speaker was Dr Mark Wilkinson whose title ‘Let’s Pwn Some Noobs’ perhaps meant more to videogaming prizewinners than to the oldies. His lively and entertaining presentation gave the young people some background about Game Theory, and a chance to participate in illustrative activities.

Mark then presented the prize mugs, while the certificates were presented by Ruth Forrester of the Moray House Institute of Education.

The afternoon was generally felt to be a successful advertisement for mathematics, for Mathematical Challenge, and for partnership between the two universities.

We would like to say a warm ‘thank you’ to all of the teachers and university colleagues who have helped with the organisation and running of the Prizegiving, and to the Heriot-Watt School of Mathematics for generous financial support.

There is still plenty of evidence that the competition continues to be highly valued by pupils, parents and teachers, and plays an important role in encouraging the mathematicians of the future.

Eric Brown and Alex Martin

Section 4, West: Secondary Division
The total number of entries was comparable with last year, just slightly up at 442 entries, and the number of schools too at 24. There were fewer Junior entries but a lot more Middle entries. This may reflect an enthusiastic junior cohort from last year. Submissions were marked with enthusiasm by volunteer postgraduate students and staff at Glasgow who, as ever, enjoyed the creativity and diligence of pupils at all levels across the region.

Our prize giving was held on 7th June, as last year. It’s always hard to choose a date which suits all schools but attendance was strong with over 80 prize winners, teachers and parents. This year we had to camp out in the Kelvin Building at Glasgow University owing to larger meetings taking place in the School of Mathematics & Statistics. Mugs and certificates were distributed and sent on later to those unable to attend. The speaker was Chris Smith of Grange Academy. Chris was last year voted Teacher of the Year and he gave us an entertaining account of activities undertaken on ‘Pi-day’ (March 14th) by pupils at Grange over the last six or seven sessions.

Chris Athorne

Section 4, West: Primary Division
The number of participating schools is still in decline despite my best efforts to increase participation. The award ceremony was held at the University of Strathclyde on Friday 7th June 2019 in a much smaller venue than usual – the grandly titled Robertson Wing of the Arbuthnott Building! Attendees came from far and wide – I was particularly pleased by the efforts of the Islay posse who once again made their annual pilgrimage to the ceremony from Port Charlotte and Port Ellen.

Although we didn’t know it at the time, we had a soon-to-be-nationally-famous winner of BBC’s The Family Brain Game – Heidi Smith and her family. You may have heard of her dad – shy and quiet maths teacher in Ayrshire....
The topic of the day was mathematical paradoxes (in honour of that very difficult courgette question) and in particular The Birthday Problem. A couple of twins on the first row provided me with a satisfactory demonstration of this famous paradox, and they were born in January which made it even better!

My warmest thanks to my colleague Amanda Corrigan for awarding the prizes and also my prospective probationary maths teachers Lienne Brown, Hannah Gillan, Stephen Hodge, Louise MacDonald, Danielle Martin, Lucy Mowat and Oscar Wilson for all of their help and support on the day.

John Winter

Enterprising Mathematics in Scotland: The 2018 Final

James Burns

The 2018 Enterprising Maths in Scotland final, sponsored by The Institute and Faculty of Actuaries, took place on Tuesday, 6th November in the exciting and modern surroundings of The Glasgow Science Centre. Some 272 Third and Fourth Year pupils in 68 teams from all across Scotland competed throughout the day in four varied and challenging rounds for the prestigious title of EMIS champions 2018.

The day began with the Team Round as contestants were faced with twelve tricky problems themed around bonfire night. Congratulations go to Bell Baxter High School for winning the special prize for the Team Round.

Round Two was a practical round where teams raced to build and answer questions on an Archimedean solid. Top marks to Beeslack High School for their performance in Round Two.

Pupils were then given some time to explore the treasures of the Science Centre during a well-earned break.

After a break, teams reconvened for the Relay Round, an action packed end to the day that was witnessed by Deputy First Minister and Education Secretary John Swinney and saw Perth High School emerge as round winners.

The scores across all four events were compiled and our congratulations go to Inveralmond Community High School who finished in 3rd place overall.

After lunch teams tackled the Stations Round where they worked against the clock on six fiendishly difficult challenges which required teamwork, logical thinking and perseverance in order to succeed. Well done to George Heriot's School who took the special prize following their outstanding performance in this round.
Congratulations also to Hutchesons’ Grammar School who finished a close second.

This year the EMiS silver salver for 1st place went to the well deserving team from James Gillespie’s High School.

EMiS would like to thank The Science Centre for hosting the event, Deputy First Minister and Education Secretary, John Swinney, for taking time out of his busy schedule to address teachers and young people at the end of the day and an extra big ‘thank you’ to The Institute and Faculty of Actuaries whose generous sponsorship made the whole event possible.

EMiS team for 2018:
James Burns, Cumnock Academy
Jane Creevy, Douglas Academy
Susan Lynch, Holy Rood R.C. High School
Siobhan Martin, St Thomas Aquinas R.C Secondary
Chris Smith, Grange Academy
Laura Ross, The Scottish Government
Peter Todd, Arbroath High School

MORE PUZZLES FROM PIE

The best puzzles from recent years’ issues of Mathematical Pie have been chosen for this new book. Some of them are quite easy, but by no means all are. Anyone of almost any age who has an interest in mathematical puzzles will find something in this book to interest and challenge them.

MA Members £7 Non Members £10
To order your copy telephone 0116 2210014 email sales@m-a.org.uk or order online at www.m-a.org.uk
The Scottish Mathematical Council Achievement Award was instituted in 2016 as a way of celebrating the very best in mathematics education in this country. In acknowledgement of their outstanding, career-long contributions to the learning and teaching of mathematics in Scotland, the first three awards went to Adam McBride, Clive Chambers and Maureen McKenna. The recipient of the award for 2019 was Brenda Harden. The presentation was made at the SMC Conference in Stirling University in March by the outgoing SMC Chair, Chris Pritchard. The photograph below shows, from left to right: Chris Pritchard, award-winner Brenda Harden, Carol Lyon, Heather Reid, and Stephen Watters, organiser of the SMC Conference for his fourth and final time.

The citation reads:

Brenda’s career began in Ellon Academy in 1978; she moved to Mackie Academy as APT in 1990, then to Turriff Academy as PT in 2005 and to Northfield Academy as Faculty Head in 2010. She finished up with a brief spell as one of the Depute Heads of Torry Academy, though she is still doing some lecturing on the Initial Teacher Education programme at Aberdeen University.

This last role seems particularly fitting, because one of Brenda’s key strengths is getting the best out of her colleagues and developing young talent. In fact, she was nominated for this award by two of her young colleagues at Northfield Academy, backed by a member of the school’s SMT. They describe her as an inspiration; “we feel we are learning from the best”. “She is so committed to her job, we joke about how she must sleep in the cupboard in her classroom instead of going home.”

They wrote about Brenda’s annual organisation of hands-on activities for S2 pupils, her involvement in Aberdeen’s Mathematics Masterclasses, how she wows her pupils and colleagues with her knowledge of number properties and mathematical facts and tricks. They wrote that Brenda is “the most passionate person about maths we have met” and her SMT link described her simply as “the Maths Guru” to whom everyone turns to for advice. “Her energy both physically and mentally is to be admired.”

In Aberdeenshire and Aberdeen City, Brenda has been involved in the Numeracy Strategies Groups, chaired the Mathematics Network Groups and helped organise five local conferences. She helped organise Maths in the Pipeline and STEM in the Pipeline; she worked in various capacities for SQA and developed new courses with Education Scotland. She has also been an integral part of the SMC’s Mathematical Challenge for almost two decades, as marker, committee member and now Secretary. And she has just completed six years on the Scottish Mathematical Council, much of that time leading us towards financial good health as our Treasurer.

Brenda Harden is a truly outstanding teacher and the well-deserving recipient of the 2019 Scottish Mathematical Council Achievement Award.

*Chris Pritchard*
Last year, we reported that Grange Academy won the *Mathématiques sans Frontières* competition, holding off all competitors with a fantastic submission from their senior team where they scored 97.7%. All eyes were on them, and their Family Brain Game winner teacher Chris Smith, to see if they could retain their title this year...

The competition ran as before, with 10 questions for Junior teams and 13 questions for Senior teams, one of which is posed (and must be answered) in a foreign language. This year, we welcomed 79 entries from 43 schools across Scotland, from as North as the Isle of Skye (Portree High School) to as South as Dumfriesshire (Douglas-Ewart High School, Kirkcudbright Academy, St Joseph’s College). This was almost double the number of entries from last year! We were glad to welcome five entries from Giggleswick School in England and also glad to see that Cumbernauld Academy once again kept their competitors energised with a smörgåsbord of European foods (Muir, 2019)!

*Mathématiques sans Frontières* was originally created by the Académie de Strasbourg in order to forge links with neighbouring countries, promote mathematics and modern languages, and to promote teamwork between pupils of all abilities. It is of no surprise then, that the high performing entries are often from those classes with a clear game plan, as can be seen in Figure 1.

The best answered question in this year's competition concerned sweets:

I distributed 100 sweets into 5 boxes. The number of sweets in each box contains the digit 8. Only two boxes have the same number of sweets. How many sweets are in each box?

Only five teams didn’t manage to score full marks on this question. On the other hand, only ten teams achieved a perfect score on this question concerning tile patterns:

Gallo-Romans made square tile patterns. To create a square with sides measuring 70 cm, they used 5 kinds of pieces namely:

- 4 squares each with side measuring 25 cm;
- 4 triangles (not isosceles) each with a hypotenuse measuring 25 cm;
- 2 isosceles triangles with the equal sides each of length 25 cm;
- 2 more isosceles triangles with the equal sides each of length 25 cm;
- 1 rhombus (not square) with each side of length 25 cm.

Using a scale of 1:5 draw a square showing how these 13 pieces could be put together:

Mearns Castle High School once again entertained the markers with some original artwork (Figure 2), and last year’s Junior winners, Girvan Academy, provided some well-constructed solutions which demonstrate exactly what the markers are looking for (Figure 3).

It was great to see so many new schools submit entries this year, and all were invited to the Top 10 Prize-giving, held at the University of the West of
Scotland’s new Lanarkshire Campus. Pupils were treated to branded pens, chocolates, and (subjected to) a short lecture on possible careers involving Mathematics from Dr Alan Walker (UWS).

The top performing entries for this year can be seen in the panel below. There were new Top 3 entries in the Junior competition from Wellington School, Grange Academy and Caldervale High School. Further, there were new Top 3 entries in the Senior competition from Robert Gordon’s College and Girvan Academy.

Overall Winners
Grange Academy: Mr Smith’s Seniors

Top Junior teams
1. Wellington School
2. Grange Academy: 3M1
3. Caldervale High School

Top Senior teams
1. Grange Academy: Seniors
2. Robert Gordon’s College: RGCAP
3. Girvan Academy

However, the outright winner of this year’s competition was Grange Academy’s Seniors entry, who improved on last year’s score with an unbeatable 100%. Their current juniors have some big boots to fill next year! As a reward, pupils from Grange Academy were treated to a full day visit from the Happy Puzzle Company (https://puzzlechallengedays.co.uk/). Pictured are two of the pupils from the winning team (and teacher Chris Smith), with their old and new shield trophy shields, during the Happy Puzzle Visit.
The competition couldn't run without the hard work of Robert D., Robert L., Gabriella, and Sandra (North Lanarkshire Council), our translator Wan (UWS), and the team of volunteer markers: Alan, Chris, Jonathan, and Wan. Thanks go to them, to the Scottish Mathematical Council who provide funding for prizes, and to UWS for hosting the prize-giving event. Finally, we thank the teachers and pupils who engage so well with the competition!

If you are interested in learning more about Mathématiques sans Frontières, would like to join our marking team, or wish to enter next year's competition, please contact Dr Alan Walker (alan.walker@uws.ac.uk), visit the website https://tinyurl.com/yveb3yyu, or see us on Twitter at www.twitter.com/MSF_Scotland.

Reference

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People, Places, Practices
Joint BSHM - CSHPM/SCHPM Conference
University of St Andrews, Scotland
6th - 8th July 2020

People, Places, Practices, is the 5-yearly joint conference of the British Society for the History of Mathematics and Canadian Society for History and Philosophy of Mathematics/La Société Canadienne d'Histoire et de Philosophie des Mathématiques, in collaboration with HOM-SIGMAA, the History of Mathematics Special Interest Group of the MAA. All are welcome to the conference. You need not be a member (though registration rates are lower for members) The conference is hosted by the School of Mathematics and Statistics, St Andrews University, the home of the MacTutor History of Mathematics Archive.

An Education Strand within the conference will run on 6-7 July. This will provide practical talks and workshops for those teaching the 15+ age group. Professor Évelyne Barbin, author of Let History into the Mathematics Classroom will talk about the French experience, where history of mathematics has recently been made a required part of the secondary mathematics curriculum. The programme will be designed to minimise accommodation costs (0 or 1 night) and we are exploring ways of enabling remote participation.

The Scottish Mathematical Council is delighted to be able to offer bursaries to support attendance of teachers in Scottish schools (secondary or primary) at this Conference. Funding of up to £200 is available for each teacher who submits an application letter to the SMC, with consideration of the following guidelines:

1. The bursary can fund travel costs, registration fees, or child care cover costs, accrued during the attendance of the conference. It cannot fund salary or related costs.
2. Candidates must provide a supporting statement from a line manager. The reference will seek to provide support, as appropriate, to how the applicant will benefit academically and professionally from this conference, how the applicant will feedback from the conference, and the applicant's need for the bursary funding to enable attendance.
3. The Scottish Mathematical Council welcomes applications from aspiring teachers at the outset of their career and from more experienced teachers working in any phase of education.
4. A condition of award is that successful applicants will produce a report on their experience, suitable for publication in the Scottish Mathematical Council Journal.

Submissions should be sent to Mr Simon Fogiel, SMC Secretary, at s.fogiel@rgc.aberdeen.sch.uk.
This article shares how staff have created a cluster-wide maths newsletter and also how pupil ambassadors (pictured above) are supporting the learning and enjoyment of mathematics.

Promoting communication in our cluster

We have all been there, a cluster in-service to discuss progression in numeracy. Then you find out that you can’t hide and sit with your friends; senior management have organised groups to ensure there is a range of staff at each table, and you don’t know any of your group. These groups are rarely as productive as SMT would like, and when they do get round to talking about progression it can become a bit of a blame game.

We have always had a keen interest in enhancing the links between the secondary school and our feeder primaries but have struggled to do anything of lasting impact, due to time, and our issue of having two identified cluster schools, but an intake from over twenty schools. A common course was discussed when Curriculum for Excellence was introduced. However, since so many of our S1 pupils don’t come from St Andrew’s or St Teresa’s, it never developed into anything more than an idea. So we, like many others, paid lip service to transition activities and ran one-off events.

What we wanted to do was create a way for staff in the three schools to communicate more freely. During maternity leave in 2018, the second author did a lot of professional reading and spent a fair bit of time thinking about how her 3-year old was learning at nursery compared to the pupils in her classes at school. She wanted to find a platform to share this with colleagues, and so the idea of the termly Maths and Numeracy Newsletter was born.

The aim of the Newsletter is to increase professional dialogue in a positive way by sharing a snapshot of learning that has happened across each level from Early through to Advanced Higher. Teachers are invited to submit a small paragraph that explains what the pupils have been learning and the type of resources they have accessed to do this, along with a photo of a pupil’s work. Some teachers include the E&Os covered and links to other curricular areas.

Staff across all three schools have a wealth of knowledge and experience that we felt could help fellow colleagues. What we find, however, is that they will rarely talk positively about what is going on in their rooms as they do not want to be seen to be boasting.

In August 2018 we sent out an invitation to staff to be involved in this project, together with a template of how the newsletter would look, to help get them on board. St Andrew’s Primary has a school blog which they update regularly with pupils’ learning and experiences, so this newsletter didn’t seem like extra work to them. Slowly, the newsletter started to take shape as it was populated with work from across the schools. We also decided it would be a good place to share ideas for CPD events, podcasts
and reading for staff to engage with. We received a lot of positive feedback but also some ideas on how to improve it. We were reminded that with Numeracy being a responsibility for all in the Broad General Education phase, we could be sharing the newsletter with all staff, not just the maths and primary staff. We now do this and have a page for Numeracy across the curriculum where a number of different departments are keen to share work that they are doing with pupils.

Although it was suggested that the newsletter should be shared with parents, we decided that this is already done through the main school newsletters for extra-curricular activities. We wanted to build trust between staff to promote moderation. It was felt that the inclusion of parents may lead to staff not wanting to participate in the project. They may also feel extra pressure to produce an outstanding snapshot and to focus on the presentation for parents.

Communication between the schools has certainly improved over this short time, with many discussions about how we can better support each other in the future through the sharing of resources and expertise. Next year we are hoping to have a focus for each term on a barrier to learning in maths with suggestions for how staff can support pupils.

**Senior Phase Ambassadors**

As well as linking with the primary staff we like to encourage our senior pupils to take on leadership roles through the maths ambassadors programme. Our ambassadors support staff by producing:

- newsletter articles and notices for pupil bulletins and social media updates
- S1/S2 maths puzzles
- homework groups at lunchtime
- monthly maths challenges for P6 to S4
- Pi Day activities
- the national maths challenges
- the maths awards event in June.

Pupils from the senior phase are invited to apply to be an ambassador. They do not have to be the highest achieving in maths because it’s more of a leadership role but they must have a passion for the subject. Through the application form the pupils have to answer the following questions:

- What do you enjoy about mathematics?
- What qualities do you feel you can bring to the maths team?
- Identify examples where you have used your communication skills effectively.
- Give examples of how you have used your leadership skills or worked effectively to help others.

They are also asked to indicate which ambassador role interests them the most. For example, the pupil...
who writes our school newsletter article would like to study journalism at University.

The ambassadors have taken some pressure off staff to run after-school clubs for the junior years, as we currently have maths staff staying behind four nights a week running sessions for National 5, Higher and Advanced Higher. The support with running maths challenges has allowed us to enter more competitions and celebrate the creative element of maths.

Like many schools, St Joseph's has a house group cup where points are mainly gained through sporting events. With the help of the ambassadors we designed an S1 maths challenge as a great way to add to the academic side of the house group points. We used the Famous Mathematician E&O for the S1 challenge, as it’s the one that is probably not afforded a lot of time in our curriculum. Pupils were split first into house groups and then in to mixed-ability groups. Due to our numbers in S1 we had to run the event twice. Pupils were told immediately after Christmas which team they were in (these were put together and discussed with the maths department and Pupil Support to ensure there would be no major issues). At this point the teams were told what mathematician they had to research for Round 1, the poster competition. They were given team work points for completing this. One team went as far as organising a meeting and creating a plan for their poster!

Round 2 consisted of 10 questions to solve together with the support of the ambassadors. We selected these questions from the United Kingdom Mathematics Trust (UKMT) Junior Mathematical Challenge papers and some of the other easier UKMT questions to ensure the challenges were accessible to all. Although the ambassadors were unable to attend the S1 events due to timetable commitments, their help in the organising and the marking of the poster and group rounds was invaluable.

Through both of these projects we are breaking down stereotypes surrounding maths and showing both students and staff the value and importance of this subject in society. We have been looking to introduce Ambassadors to the Primary context. Maths is becoming more accessible and the creative element is being celebrated, which when introduced at Primary level will have a lasting affect when transitioning to secondary. If Maths Ambassadors are introduced earlier, it may have an impact on pupils’ outlook and interest in maths further into their school careers.

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**Mathematicians on Coins (1)**

Mathematicians and their work feature on many stamps; check out Robin Wilson’s talk on the subject at www.youtube.com/watch?v=cpqJIF3WeXA. Mathematicians on coins do not fare so well. Here are some of the biggest hitters in our subject’s development.

![Pythagoras](image1.png)

**Pythagoras**
Greece, 2013

![Pythagoras](image2.png)

**Pythagoras**
Uganda, 2000

![Archimedes](image3.png)

**Archimedes**
Greece, 2015

![Archimedes](image4.png)

**Archimedes**
Njue, 2014

![Isaac Newton](image5.png)

**Isaac Newton**
UK, 2009

![Isaac Newton](image6.png)

**Isaac Newton**
Tristan da Cunha, 2017

![Gottfried Leibniz](image7.png)

**Gottfried Leibniz**
West Germany, 1966

![Gottfried Leibniz](image8.png)

**Gottfried Leibniz**
East Germany, 1966

![Carl Friedrich Gauss](image9.png)

**Carl Friedrich Gauss**
East Germany, 1977
The stimulus for this short note was a talk given by Liz Meenan at a Mathematical Association Annual Conference a couple of years back. A consultant in mathematics education, Liz has a penchant for the mathematics of paper folding and she started by showing the audience how to produce a 'Cairo tile' in the form of an isosceles pentagon by folding a strip of paper just once.

The particular strip which she took is that which is left over when the largest possible square is removed from an international A-sized sheet of paper. Here we let the sheet have unit width and length (height) $\sqrt{2}$. Then the dimensions of the strip are $\sqrt{2}-1$ and 1.

Take the strip and fold it so that diagonally opposite corners meet. This indeed gives an isosceles pentagon consisting of an isosceles triangle and two right-angled triangles:

![Diagram](image)

Taking one right-angled triangle, we note that one of its shorter sides is $\sqrt{2}-1$ and the lengths of the other two sides sum to 1. So if we let the hypotenuse have length $x$ and the third side length $1-x$, then by Pythagoras’ Theorem:

\[
x^2 = (1-x)^2 + (\sqrt{2}-1)^2 = 1 - 2x + x^2 + 3 - 2\sqrt{2}
\]

\[
2x = 4 - 2\sqrt{2}
\]

\[
x = 2 - \sqrt{2}
\]

So, $1-x = \sqrt{2} - 1$, and hence the triangle is isosceles with two angles of 45°. And since the strip was rectangular before being folded, all three angles at the top are 45° and this fixes all the angles in the pentagon.

We can also see the pentagon as part of a regular octagon:

![Diagram](image)

The tiles referred to as 'Cairo' are indeed pentagonal and isosceles and so this is an example of such a tile. But note that neither the tile in the article by Andrew Jobbings in *SMC Journal* 47, nor that in my article *Journal* 46, is the same as this one. (They have angles of 120° and 144° respectively at the top.) However, as with all Cairo tiles, when arranged in clusters of four they give a hexagonal super-tile which it is easy to see tessellates the plane:

![Diagram](image)

**Author**

Dr Chris Pritchard, contact details on p1.
It's a real shame that we Secondary teachers don't get enough opportunities to blether to very young kids. Every time I speak to groups of wee pupils I'm reminded that these guys are the Mathematicians of tomorrow. And although the word “Maths” usually means nothing to a bunch of nursery pupils, as soon as we mention numbers or patterns or shapes the room is buzzing with enthusiastic 4-year-olds, desperate to show off their counting skills and tell you their expert knowledge of triangles.

If we can harness this sort of wonder with pupils further along their journey through school then it could be a powerful motivator. This rationale inspired a new resource I've created called “Wow, How, Now, Take a Bow”. This series of lessons have been put together with bright upper-primary pupils in mind but in the hands of a skilful teacher could be used with S1/2 classes and younger kids too!

The four-strand format is simple:

1. capture students' excitement with a "wow" moment (either a Maths-based trick or engaging activity)
2. explore "how" the trick works (thinking about the Maths underpinning the trick)
3. discover a way these concepts are used "now" (modern-day applications of Maths, particularly in STEM)
4. permit noteworthy Mathematicians to "take a bow" (learn a bit about relevant, interesting stories from the history of Maths).

The theme for my first three samples are unintentionally alliterative but explore three big Mathematical concepts: Primes, Position and Proof.

The resource is available to download for free from TES [www.tes.com/teaching-resource/wow-how-now-take-a-bow-maths-12084236]. I think the concept is flexible enough to be de developed further and I might add more in the future.

Hopefully these ideas will spark a bit of Mathematical excitement, enthusiasm, intrigue and vibrancy into your classroom!

Pupils from James Gillespie’s High School are introduced to the concept of proof as they get their teeth into McNugget Maths at the launch event for Maths Week Scotland 2019 at the National Museum of Scotland

Chris Smith
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1. Type a three-digit number into your calculator. For example, 314.

2. Re-type that three-digit number into your calculator (so you’ve now got a six-digit number).

3. Divide your six-digit number by 13.
4. Now divide your answer by 11.
5. Finally divide your answer by 7.

How does this trick work? To explore this...

1. Think about the process of dividing by 13 then 11 then 7. What single number could we have divided by instead?
2. Write out the 1001 times table from 1001 x 100 up to 1001 x 115. What do you notice?
3. How can we tie together these two neat Maths facts to make this "trick" work?

How many numbers divide exactly into 15?
There are four: 1,3,5 and 15 (these are called the factors of 15).
The number 16 has five factors (1,2,4,8 and 16).
The number 17 only has two (1 and 17).

Numbers that have exactly two factors (like 17) are called prime numbers. The first few prime numbers are 2,3,5,7,11,13,17,19,23,29,...

Counting numbers are either prime numbers or a combination of prime numbers multiplied together. For example...
38 = 2 x 19 (so 38 isn’t prime)
39 = 3 x 13 (so 39 isn’t prime)
40 = 2 x 2 x 2 x 5 (so 40 isn’t prime)
41 is prime
42 = 2 x 3 x 7 (so 42 isn’t prime)
43 is prime
44 = 2 x 2 x 11 (so 44 isn’t prime)
45 = 3 x 3 x 5 (so 45 isn’t prime)

Prime numbers are the building blocks of the number world and there are infinitely many of them.
Prime numbers are used nowadays to encrypt important information (each time credit card details are used on Amazon, prime numbers are used to keep the transaction secure!)

Back in the 1990s, a Nicely named Mathematician was doing some Maths at home. It involved prime numbers.

Thomas "accidentally" found a computer bug in IBM’s new Pentium Chip which led to errors in quite basic calculations. All computers with the faulty chips had to be fixed, costing the company about $500 million.
Your teacher will show you four squares of card with numbers from 1 to 16 on each side and some sections cut out:

They'll ask you to pick a number (from 1 to 16) but to keep it a secret...all you tell them is whether your number is the right way up. If you tell them it's upsidedown they'll rotate the card. One by one the cards will be placed on top of each other until the fourth card (the one with no sections cut out) is placed on top of the others. You might get a surprise when the cards are turned round and see what appears at the back.

The fourth card had the numbers 1 to 16 on the back. Make your own set of cards and practise the trick a few times to get your head round what's going on!

But how did this trick work? How did the arrangement of the cards lead to a unique number being revealed?

How would you construct a more basic version?

How many cards would you need if we're just using the numbers 1 to 4 or 1 to 9?

The positioning of the numbers on the cards was crucial.

The Maths of position, shapes and movement is used in movie special effects and animation.

Animation teams, like the ones at Pixar Studios, use Maths all the time to make Sulley roar when he's scarying in *Monsters Inc.* or to allow Merida to shoot that perfect arrow in *Brave* or Nemo to glide through the water finding *Dory*.

Pixar's Senior Research Analyst Tony Derose explains how important the Mathematics of position is to their incredible films in a TED talk:

www.youtube.com/watch?v=_lZMYMf4NQ0

There's a story told about French Mathematician René Descartes. He was lying in bed watching a spider crawl across the roof tiles and he devised a grid system for describing the spider's position.

This grid system is called the Cartesian Coordinate system after the genius Descartes.

... Rene Descartes
McDonald's originally sold their delicious Chicken McNuggets in boxes of 6 nuggets, 9 nuggets, or 20 nuggets.

So you could buy twelve nuggets (two boxes of 6) or 29 nuggets (a box of 20 and a box of 9) but you couldn’t buy three nuggets or 10 nuggets.

Work out all the numbers of nuggets that you can buy from 1 up to 50. Which numbers are missing?

There is actually a highest number of nuggets that you can’t buy!

Can you find it? How do you know it's the highest? Can you prove it?

I know what you’re thinking. What about a Happy Meal? One of those allows you to buy a box of four nuggets. So now we can use 4, 6, 9 and 20.

How does that change things? How can we find some of the missing numbers we found earlier? How can you prove that your new impossible nugget number is the highest?

This "McNugget Problem" is also called the "Frobenius Problem"!

"Proof" is something that Mathematicians are obsessed with - they want to be able to know for sure whether a statement is always true, sometimes true or never true...this can be hard!

Here are three simple "rules". Do you think these are always true, sometimes true or never true?

A: If you add an even number and an odd number the answer is an even number.
B: Doubling a number makes it larger.
C: Adding three consecutive integers gives a number that is divisible by six.

For centuries Mathematicians have known that you could find pairs of square numbers that add up to another square number!

—for example, $3^2 + 4^2 = 5^2$ (9+16=25) and $9^2 + 40^2 = 41^2$ (81+1600=1681).

Can you find any others?

In the 1600s a lawyer called Pierre de Fermat said that he could prove that square numbers are special and you'll never find any a similar thing for any powers bigger than two. But he didn't write this down (he blamed the margin in his book being too small)!

Over 350 years passed before Andrew Wiles became a Mathematical hero and proved Fermat's Last Theorem!
Introduction
When I ask teachers what aspects of numeracy their pupils need to improve the most, they often say place value or division. When I dig deeper, the problem usually lies with the fact that the same pupils are also struggling to understand multiplication. They may be able to recite their tables but have little understanding of how the values are related. At best, they can use repeated addition to write out any times table but this is often very time-consuming.

Making the links
The Glasgow Counts Framework for Numeracy and Mathematics encourages teachers to take a problem-solving approach, helping pupils relate what they are learning to a familiar context. We use concrete resources to help learners visualise the actions that are taking place, making sure that informal pictorial representations directly link with these actions. The will be sharing out items. These actions still need to become larger.

Problem: 78 cakes are shared between 6 friends. How many cakes do they each get?

Sharing out 78 individual counters here would be inappropriate. Place value counters and base 10 blocks encourage learners to group in tens, exchanging a ten for ten ones when needed. When sharing the resources learners naturally partition 78 into (66 + 12) or (60 + 6 + 12) They discover the need to exchange the ten for 10 ones to enable them to continue sharing. They also see that 6 lots of 13 equals 78, and 78 divided into 6 equal shares is 13, relating multiplication and division as inverse processes (see Fig. 1).

These actions can be illustrated as an expanded algorithm:

\[
\begin{align*}
6 \div 12 & = 13 \\
6 \div 10 & = 6 \div 60 + 6 + 12
\end{align*}
\]

alternatively,

\[
\begin{align*}
6 \div 60 & = 6 \div 4 = 13 \\
6 \div 60 + 18 & = 6 \div 10 + 3
\end{align*}
\]

Such steps are essential before moving onto the traditional, formal algorithm.

Taking time to work with concrete resources helps develop a deeper conceptual understanding of place value. Contrasting the actions involved in sharing with the language we traditionally use, it is hardly surprising that learners’ understanding of place value can be limited. Should our language not be

70 makes 6 groups of 10, with 10 left over. This ten is carried into the ones column creating 18 ones

rather than,

6 goes into 7 once, with 1 left over.
6 goes into 18, 3 times?

To help learners move on from using concrete resources we encourage them to list the building blocks of the multiplication table which we call the VIPs (very important products). The original problem of ‘78 cakes are shared between 6 friends. How many cakes do they each get?’ is rephrased as ‘6 lots of what is equal to 78?’ The VIPs are built simply by repeated doubling:

\[
\begin{align*}
6 \times 1 & = 6, \\
6 \times 2 & = 12, \\
6 \times 4 & = 12 \times 2 = 24, \\
6 \times 5 & = 12 \times 2 \times 2 = 30, \\
6 \times 10 & = 60
\end{align*}
\]

while calculating 6 \times 5 is carried out by multiplying by ten (6 \times 10 = 60) then halving to give 30.

\[
\begin{align*}
VIPS & \\
\times 1 & = 6 \\
\times 2 & = 12 \\
\times 4 & = 24 \\
\times 5 & = 30 \\
\times 10 & = 60 \\
\end{align*}
\]

This could also be shown on an open number line. Alternatively, using base 10 materials encourages partitioning into tens and ones, and is a visual representation of the popular area model/grid method (see Fig. 2).

Making the link with remainders
For many of our learners, making the jump from identifying remainders to expressing an answer as
a decimal can be very confusing. We have found that this leap can be made easier if we consider linking to our existing knowledge of fractions. Smaller steps are needed to ensure a fuller understanding.

**Problem:** 78 cakes are shared between 4 friends. How many cakes do they each get?

Again, learners may wish to use place value counters to help make the actions visible and aid the partitioning of a large number (see Fig. 3).

When sharing the counters learners partition 78 into \((48 + 30)\) or \((40 + 8 + 30)\). They understand the need to exchange the 3 tens for 30 ones to enable them to continue sharing. Although time-consuming and sometimes frustrating, this experience is essential to embedding a deep understanding of the actions taking place. The question is how to deal with the remainder; can it be shared equally? In this example, two cakes can be shared equally with each person getting either two quarters or one half. This leads to rich discussions about equivalent fractions.

For every problem they tackle, we encourage our learners to ask themselves, 

*How can I make this problem easier? How many different strategies can I use to solve this problem?*

Dividing by 4 can be linked to their knowledge of making quarters. The easiest way to make quarters is to halve and halve again. Partitioning into even numbers will ease dividing by 2.

\[
\begin{align*}
78 &+ 4 \\
&= (60 + 18) + 2 \\
&= (30 + 9) + 2 \\
&= (30 + 8 + 1) + 2 \\
&= 15 + 4 + \frac{1}{2} \\
&= 19 \frac{1}{2} \\
&= 19 \frac{5}{10} = 19.5
\end{align*}
\]

For this method there is no need to consider a remainder as the answer is left as a fraction. We can also convert our answer to decimal form by first converting the half into tenths. When we first introduce decimal fractions we read 19.5 as “19 and 5 tenths”. This helps pupils’ understanding of place value as there is a five in the tenths column. Later, when learners have demonstrated a clear understanding of place value, this can be read as “19 point 5”.

Following the steps from sharing, to division, to fractions, to decimal fractions, gives learners a logical pathway. This allows them to make links across different aspects of numeracy.

As the level of numeracy becomes more sophisticated, we encourage learners to consider re-imagining the problem in a different context.

**Problem:** £78 is shared between 5 friends. How much money do they each get?

Some learners will still benefit from using concrete resources (see Fig. 4). Others will already have a good grasp of multiplication facts and be able to progress straight to the expanded algorithm.

<table>
<thead>
<tr>
<th>VIPs</th>
<th>578</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\times 1)</td>
<td>5</td>
</tr>
<tr>
<td>(\times 2)</td>
<td>10</td>
</tr>
<tr>
<td>(\times 4)</td>
<td>20</td>
</tr>
<tr>
<td>(\times 5)</td>
<td>25</td>
</tr>
<tr>
<td>(\times 10)</td>
<td>50</td>
</tr>
</tbody>
</table>

Using the context of money leads to rich discussions around equivalence. By sharing out coins instead of place value counters, learners can see that 20p is one fifth of a pound, a tenth is half of a fifth, and three fifths and six tenths have the same value. Playing with coins also leads to rich learning experiences around partitioning and adding fractions. Omitting the pound symbol we have:

\[
\begin{align*}
\frac{1}{10} &= 10p \\
\frac{6}{10} &= 60p \\
\frac{1}{5} &= 20p \\
\frac{3}{5} &= 60p \\
60p &= 50p + 10p = \frac{1}{2} + \frac{1}{10} = \frac{5}{10} + \frac{6}{10}
\end{align*}
\]

The aim of this article is to provide opportunities for pupils to see what division in action looks like, leading to a deeper understanding of the concept. Using the suggested approaches should help clarify mathematical thinking.

---

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Immediately following school I gained an Honours degree in Economics and started work within the Public Sector. In order to progress my career I decided to return to full-time education and secure a Masters degree in Statistics. I enjoyed working as a Statistician within the housing and renewable energy sectors for 12 years but came to the realisation that I needed a more challenging and personally satisfying role. In order to pursue the goal of becoming a Maths teacher I took the decision to give up my career in statistics and enter full-time education to complete a BSc in Mathematics.

I had always really enjoyed working with, and solving problems with, numbers and believed that becoming a Maths teacher would enable me to make a more valuable contribution to my community. The satisfaction and enjoyment gained by completing a difficult puzzle, or the sudden realisation of a concept clicking into place, was something that I thought I would enjoy sharing. With enough motivation and support I believe anyone can succeed at Maths. I wanted to inspire and motivate the next generation to be as good as they could be at Maths (and better than they thought they were capable of). We use Maths in every aspect of our lives at work and in practical everyday activities at home and beyond. Decisions in life are so often based on numerical information. Now the digital age presents us with more numerical data than ever before and puts a new premium on numeracy skills.

Whilst completing my Mathematics degree I spent one day a week, over an 18-month period, volunteering as a classroom support assistant in the Maths department of a High School. This gave me valuable experience within the classroom environment and enabled me to observe the different methodologies being employed by the teachers. Often working with students who were struggling with the material on a one-to-one basis, I found that breaking the tasks into more simple steps made them less daunting for the student.

On completion of my degree I was offered a place on a Professional Graduate Diploma in Education: Secondary (PGDE) course at the university nearest to where I live, but this would have meant commuting and completing my training in a region over an hour’s travel from my home. As a mature student with an established family home, whilst this was eminently achievable, it would have provided practical challenges. Fortunately I discovered that the University of Highlands and Islands (UHI) had recently established a ‘distance learning partnership’ with my Local Education department. Through this partnership UHI were providing their PGDE (Secondary) course which enabled me to attend a local college campus and complete the school placements in high schools within my local region. This not only helped in reducing travel time but also enabled me to meet, work and establish relationships with the community of teachers within the schools closest to my home.

I completed a total of 17 weeks of teaching during 3 school placements with an additional initial 2 weeks of observation. The classes that I prepared for and taught started at 0.4 FTE which was approximately 9 hours a week of actual teaching rising to 0.8 FTE, 21 hours, in the final 6-week placement. I enjoyed the time in class and found the whole ‘placement’ experience to be a very positive and valuable one. The support provided by the mentors in each school and the wider Mathematics departments was generous, with copious amounts of useful advice and resources being provided.

My initial two-week placement allowed me to observe five Mathematics teachers in classes across all year groups (S1-S6). By reflecting on the teaching practice used by these experienced professionals I was able to identify a number of methods employed to create a classroom environment that supported effective learning. These included actions to address: classroom management, relationship building and disruptive behaviour management.

A number of general but important points were provided by the Maths teachers in this placement, which included the requirements:

- to have pre-planned and clear aims/objectives for each lesson
- to ensure that each lesson delivered incremental steps that built on previous work and that could be identified by pupils as clearly and logically progressing from the previous lesson
• to deliver lessons which, whilst part of a cohesive journey, must be varied and where possible employ ‘active learning’
• to maintain a consistent and ‘firm but fair’ approach to discipline within the class and specifically, as a new teacher, to set clear boundaries from the offset.

My experience led me to the clear conclusion that success as a Maths teacher required a great deal of planning. The clear identification of success criteria and learning outcomes for each lesson, the understanding of the ‘curriculum journey’ and the incremental steps required that enable students to build their knowledge and understanding are all required to deliver effective lessons. I have been able to identify, by observing what appears to be a common lesson structure, the positive impact on achieving effective learning that is delivered by using a structure of a ‘starter’, followed by the ‘core’ lesson, followed by a ‘plenary’ session. I was encouraged to see a number of examples of active learning used and the enthusiastic engagement of students in these lessons. Whilst I recognised the planning commitment associated with active learning and the challenge of establishing this as a regular methodology for a class, the learning outcomes benefitted from its use in every example I observed.

I particularly enjoyed the first 6-week placement, during which I had sole responsibility for the teaching of a lower set S1 class along with sharing the teaching of an upper set S2 class, an S3 class and a National 5 class. The staff and Head of the Maths department were very helpful throughout my time at the school, providing advice and insights to developing effective teaching methodologies. I managed to develop positive relationships with the students, and I found that the ‘firm but fair’ and specifically a ‘consistent’ approach worked well. I employed active learning in a number of classes and used a ‘Carousel’ methodology with a National 5 class in my observation lesson. The ‘cost-benefit’ of the additional work required lead me to the conclusion that whenever possible active learning will engage more of the class, and in particular, is more likely to engage those members of the class who are less able and struggling.

During this placement I became aware that I needed to listen much more closely to what I said in class. Whatever concept or learning I was trying to communicate, I needed to ensure that I did this clearly and concisely and that I resisted the temptation, or my tendency, to communicate an abridged version based on my own personal understanding of the subject and topic. Therefore, my target was to become more aware of what I was saying and how it informed the concept or learning I was trying to convey.

My experience of the first placement reinforced my understanding of the requirement to think clearly, and in detail, about how Mathematical concepts and solutions are explained to a class. I found that having a single explanation was often not sufficient and that I needed to have at least two possible ways of explaining concepts and solutions. Across the S1 – S6 range I found that, whilst the majority of the students in a class might well understand the first explanation provided, in order to ensure the whole class engaged with the learning, it was necessary to have alternative ways of explaining every concept or solution. Therefore continuing to develop as wide a range of explanations as possible became a key objective as well as keeping the successful ones I had already developed.

Generally, I was much more comfortable in my role as a teacher in my second school placement and this confidence enabled me to create stronger relationships with the students I was teaching. Whilst establishing better relationships and gaining confidence in my role, I found that discipline was still the area that created the greatest challenge. However, having learnt from my experiences in school 1, I recognised the requirement to be properly ‘organised’. Preparing lesson plans and creating the materials required ensured that I understood fully what I would be delivering during each lesson. This enabled me to pay more attention to behavioural management in class which resulted in lower levels of disruption. By gaining control of discipline, for the majority of the time, and creating lesson plans where I focused on student engagement, I was able to create a positive classroom environment and engage a larger proportion of the students. I found that by clearly communicating my expectations of students I was more able to both moderate student behaviour and reduce the number of instances of disruption. The teaching environment that I was able to achieve in school 2 meant that I could also focus on the lesson structure. By using starters in each lesson and finishing with a plenary session I was much more able to assess the success of my lesson planning and the understanding of my students.

I particularly enjoyed my final placement as I was able to apply my learning from the first two placements and quickly establish control within each of my classes. My confidence within the classroom, and within the school environment as a whole, had increased which allowed me to be more adventurous with the resources employed during my lessons and my overall delivery of each lesson. I was able to employ a more interactive style, encouraging students to think about and apply the materials and enable them to discover solutions through the application of the mathematical concepts. The overall atmosphere in the school was both inclusive and supportive and I found discipline much easier to maintain. This may have been a
result of my confidence within the classroom, or the overall school culture which definitely helped in reducing disruptive behaviour. Once again, I enjoyed the support of the teaching staff who were happy to share their experience and the successful techniques they employ in their lessons. By creating specific and structured lesson plans for each lesson I was again able to concentrate on the actual material during the lesson which enabled me to practise a more interactive/inclusive style that encouraged students to investigate, question and engage with the materials.

I found my placements particularly useful and personally satisfying and think that entering teaching as a ‘mature’ student teacher is an advantage, as my confidence in the classroom was based on a broader experience of work and life. My high expectations of all learners were recognised at each of my placements, as were my efforts to find ways to engage and develop relationships with demotivated learners to help them engage in the lesson.

I initially found self-reflection on my teaching practice challenging but the support from the mentors, my tutor and other members of the teaching staff at each of the schools was invaluable and I found their advice useful in establishing an effective strategy of reflection.

The final exercise in my PGDE course was to carry out an enquiry into an aspect of pupil learning. I had recognised during my time in placement that students learn in various different ways, and that including visual, auditory and physical elements in my lessons improved overall student engagement. Therefore my enquiry was about broadening my understanding and appreciation of different student engagement mechanisms and developing the teaching ‘toolkit’ at my disposal.

My enquiry assessed three different student engagement mechanisms; Tarsia Jigsaws, Quizizz and Show Me Boards. In order to assess the levels of student engagement created by each mechanism I used the Leuven Scale of Active Engagement which rates engagement on a scale of 1 to 5.

- **Tarsia Jigsaws**, which promote group work discussion and are useful in revision, are physical puzzles that require students to join solutions to questions in a triangular form of Dominoes.

- **Quizizz**, an online resource and useful formative assessment tool requires the students to answer multiple choice mathematical questions in a competitive timed environment.

- **Show Me Boards** is again a useful formative assessment mechanism where students answer mathematical questions on a whiteboard that they then hold up for the teacher to see.

I chose a first-year mathematics class in partnership with the regular teacher to carry out this enquiry. I informed the class of the process that would be applied and utilised each of the mechanisms during a three-week period. After each class, where each mechanism was employed, an assessment of student engagement (although I recognised this is a subjective measurement) was made using the Leuven Scale by myself as the teacher, the class’s regular teacher and the students themselves.

The first-year students found all three mechanisms enjoyable and entertaining and were highly co-operative in the enquiry process.

**Tarsia Puzzles**
(rated at 4.46 on the Leuven Scale)

This encouraged collaborative working but also again created a competitive, team environment. The teacher’s observations included

- ‘all pupils were involved and engaged (also motivated – wanted to be first to complete it) – pupils referred to their jotters and also did some calculations in their jotters – all pupils worked really hard to get answers – those pupils who finished first were able to help others.’

**Quizizz**
(rated at 4.90 on the Leuven Scale)

This mechanism created an excited atmosphere with high levels of competition. Although, it also resulted in high levels of co-operation and collaborative working. The regular teacher’s observations included

- ‘pupils worked really well together to solve the problems – pupils were explaining to each other how to get the answer – misconceptions were highlighted with instant feedback from the teacher.’

**Show Me Boards**
(rated at 4.14 on the Leuven Scale)

Once again, this mechanism created an excited atmosphere although the students remained attentive and focused. The teacher’s observations included

- ‘the pupils who usually find it hard to focus were much more on task than usual – pupils were working individually but listened carefully to instructions from teacher – all pupils participated – pupils were engaged and keen to get their answers down on the boards.’

The overall ratings of each of these mechanisms were similar although, of the three, Tarsia Puzzles were marked lowest by the students. Whilst this enquiry suggests Quizizz was the most engaging
mechanism, it has also enabled me to assess the impact on the class in a wide range of factors, including collaboration, concentration, behaviour and subject understanding. Each of the mechanisms could be deemed successful since the overall appetite for Mathematics from the students in this first-year class increased.

I am convinced of the advantage of using active learning mechanisms within the classroom and this enquiry increased my confidence in their application. I now intend to increase the number and scope of the engagement mechanisms that I can use and identify the advantages and outcomes that each mechanism delivers. I used a structured approach to my lesson planning with Starters, Main Content and Plenary that are distinct and clearly identifiable for the students. This enquiry will result in my inclusion of Tarsia Puzzles, Quizizz and Show Me Boards as regular additions to this structured approach.

I now think leadership as a teacher is much less about telling students what to do but is about empowering students by clearly defining what is expected of them. I don’t think teaching is about motivating students; it’s about creating the environment where they can motivate themselves. During my placements I made specific efforts to communicate clearly what I expected of the students in my classes and what our objectives were for each lesson. I attempted to create an environment where students recognised that they could be successful, so reducing the perceived risk of failure. I was achieving this by introducing the material in manageable sections that developed incrementally and by reviewing the progress with the students at the end of each class. This allowed students to recognise their progress and increased their confidence and their engagement.

In each of the schools weekly Collaborative Assessment Log meetings enabled me to identify areas for development and provided an opportunity for collaborative interaction. These meetings were particularly useful in the third school where, since my confidence had developed, I was able to be more ambitious with my objectives. This confidence also enabled me to contribute more actively in discussions with the teaching community on policy, teaching strategies, engagement techniques, behaviour management and health & well-being.

Overall, whilst I found the PGDE (Secondary) course both challenging and demanding, it was very rewarding and ultimately enjoyable. I am now entering the world of teaching Maths, confident that I have been appropriately prepared and am as ready as I could be.

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After six years as a Maths Teacher, I thought I was safe from the fabled “one third of new teachers leave the profession within 5 years” statistic (TES, 2019). I love my job, I love my pupils, I get to talk about Maths every day, I have found my calling in life! However, there came a point in that year where I was unable to continue, at least in the same manner as I had done in my earlier years. I remain in the profession, grateful that I have been able to do so. This relied solely on learning about mental wellbeing and adapting my practice accordingly. The SMC Stirling conference provided demonstrating the Maths community’s support and thoughtfulness. I would like to offer to you a brief tour of what to bear in mind when thinking about your mental health and the mental health of others around you.

**Education can set an example**

As educators we are uniquely positioned to both receive the benefits of developed self-care, and as an added extra, as a model to our young people, the good practice for sustainably contributing in a workplace (avoiding the dreaded burnout). The teaching profession can and should be a leader in for the mental wellbeing of staff.

**Positive and practical help**

When tackling these issues it is easy to become bogged down in analysing the vast scale of the issue and rubbernecking the devastation that poor management of mental health leads to. Understanding the root cause of the problem is worthwhile and does have its time and place, but it is not here. I will focus on a positive and practical approach towards mental health and wellbeing. Even if we reach a point where there is no mental health crisis, it would still be important to regularly check in on our wellbeing and form healthy habits and routines.

**Removal of stigma**

An important factor very early on in any discussion of mental health is removing the stigma surrounding it. As a society we have come on in quantum leaps and bounds in our understanding of mental health and the sensitivity with which we “deal with” people struggling with mental health problems – but the stigma is still very real. Allow me to lead by example and share some of the health issues I have experienced myself during my time on this Earth, in this body:

<table>
<thead>
<tr>
<th>PHYSICAL</th>
<th>MENTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asthma</td>
<td>Health anxiety</td>
</tr>
<tr>
<td>Prescription glasses</td>
<td>Generalised anxiety disorder</td>
</tr>
<tr>
<td>Plantar fasciitis</td>
<td>Low self-esteem</td>
</tr>
<tr>
<td>Various aches and pains from sport and hobbies</td>
<td>Fear of flying</td>
</tr>
</tbody>
</table>

In the first column is a mix of long-term conditions which I manage and have regular check-ups on, and short-term complaints I experience as a result of my lifestyle and recent events. In the second column we can see the exact same!

The difference is that I feel absolutely no embarrassment or shame when I need to take my inhaler or refuse to take part in an activity because I’ve pulled a muscle. I can easily complain around the staffroom table at break about my bumps and bruises. Runner friends who experience the same problems will take an active interest in things I have tried to improve to overcome the niggles. I even get compliments from people when I don a new pair of glasses!

On the other hand, the fact that the second column is now in print places me in an incredibly vulnerable position, I can feel the apprehension running down my back. But I have done nothing wrong. Is it taking a risk, a professional gamble, making a courageous (or stupid) move? No, it’s nothing. If there was no stigma surrounding this issue, then I would be able to chat about this around the staffroom break
table too. So if this makes even one person feel a bit more included, represented and normal within the teaching profession, then I’ll take that as a win.

**Mental health, mental illness, mental wellbeing**

One useful frame of reference to reduce this stigma is to consider the differences between mental health, mental illness and mental wellbeing.

A lucky few of us may never experience any mental health issues but let me be clear and state in no uncertain terms – every single one of us will experience issues with mental wellbeing at some point in our lives.

*Mental health* was defined by the World Health Organisation in 2004 as ‘a state of well-being in which the individual realises [their] own abilities, can cope with the normal stresses of life, can work productively and fruitfully, and is able to make a contribution to [their] community’ (Who.int, 2019). It is worth taking a minute to reflect on what a tall order this is, and by definition how many people could be categorised as not mentally healthy.

*Mental illness*, on the other hand, usually refers to having a diagnosable mental disorder e.g. depression, schizophrenia, intellectual disabilities, or drug abuse issues. All of them fall under the umbrella of mental illness and just think about the different connotations attached to them and how society treats people experiencing these issues.

*Mental wellbeing* refers to the ups and downs of life that everyone experiences and are usually associated with life events. For example, if you are going through a difficult divorce, your beloved dog goes and dies and you suddenly get landed with an unexpected set of reports to write, all in one week, your mental wellbeing may trend downwards for a while (NHS Scotland, 2019).

Wonderfully for us mathematicians, we can whack this onto a set of axes (adapted from Tudor, 1996).

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**The best defence is a good offence**

Even on the days when your wellbeing is good there are a few practices you can bring into your life which will ensure a healthy defence.

Even on the days when your wellbeing is good there are a few practices you can bring into your life which will ensure a healthy defence.

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The NHS recommends these five pillars to maintain good mental health (nhs.uk, 2019):

- **Connect**
  - You could make a point of going to the staffroom for break every day for a week and strike up a conversation with whoever happens to be there.
  - If you’re looking for some out of school nerdy socialising then check out your local Maths Jam: somewhere in a pub near you, on the second to last Tuesday of the month, a bunch of mathematicians get together and tweet each other puzzles and other geeky fun! Lookup the Maths Jam website to find your nearest one or join in online via twitter.

- **Keep learning**
  - Every day is a school day!

- **Be active**
  - If your school has a gym, see if there is a suitable time for you to use it.
  - Help at a lunchtime or after-school sports club.
  - Start a teachers-only sports club.

- **Give to others**
  - Remember that you do this every day by going to school and teaching as best you can, you give so much to Maths and education, don’t forget it.

- **Be mindful**
  - Try a short meditation or yoga in your lunchtime or non-contact time; there are plenty of apps available or YouTube videos online.
Challenges faced by Maths teachers

In his book, *Mental Health First Aid*, Guy Winch talks about how day to day we can experience “emotional scrapes and grazes” which affect our mental wellbeing – if left untreated they can then lead on to mental illness (Winch 2014). We are great at treating common physical ailments and have a whole cabinet full of medicines, and he posits that we should consider mental health in the same way. This mind map is based on the book and links together some of the emotional pains most relevant to teaching with an emotional first aid remedy adapted to suit teachers. This represents only a tiny number of the techniques available for your mental health first aid kit, much more detail in the book!

**Everyday Emotional Scrapes and Grazes for Maths Teachers**

- **Rejection**
  - Even very small and inconsequential rejections, such as not being passed a ball during a game, can have a measurable effect on people’s wellbeing and performance in tasks.
  - In S2 a third of pupils dislike numeracy work (SSLN, 2015) and with Maths being a compulsory subject until S4 we don’t have the luxury of pupils choosing to be in our subject. As a result some resent having to be there and reject the teaching.

- **Loneliness**
  - One of the first books any teacher entering the profession will be encouraged to read is Dylan Williams’s “Inside The Black Box” named so after the black box of the classroom where our pupils go in and they come out better at Maths - to be the only adult living in that black box sounds like a rather lonely place. Trivial as it may sound to be lonely at work, chronic loneliness has been shown to have the same effect on longevity as smoking.

- **Guilt**
  - Feelings of guilt usually arise from feeling that you have fallen short of your personal standards. As teachers we have high expectations and a difficult job so it is almost inevitable that we will experience guilt. Teaching has one of the highest levels of “presenteeism” of any profession, possibly due to this feeling of guilt and responsibility.

- **Failure**
  - Its no secret that Maths is often hard to learn and hard to teach - some failure is inevitable. Failure is also a necessary part of learning, however, repeated failure and not having the means to deal with failure in a positive way can lower self-esteem, reduce confidence and lead to stress.

- **Low Self-Esteem**
  - When you stand up in front of 30 people and speak for a living it might be easy to think that you don’t experience low self esteem, but it can present in a variety of ways. For example: people pleasing, never saying no, putting the needs of others first and feelings of not being “good enough”.

**Presenteeism Fact** (Aronsson et al., 2000; Bergstrom et al., 2009)
When To Get Help

If you are worried and feel you need help with your mental wellbeing or are worried about a mental illness then please just go to your GP. It can be daunting, but you would not be the first or last teacher they had through their doors struggling.

The NHS have a very useful online Mood Self Assessment Tool which takes you through a series of multiple-choice questions and then puts a number on a scale for the severity of any depression or anxiety you may be experiencing. This is only a starting point but can be very useful to open up a conversation with your GP if you feel you need help. The Education Support Partnership have a similar tool and lots of useful information specifically tailored to teachers on their website. And most teaching unions also have benevolent funds and are able to offer all sorts of assistance. If you think you need help, then you do – so make contact with someone.

Reflections from the SMC Conference

This article comes from a workshop which I ran at the 2019 SMC conference. The SMC conference really is like no other and offers an amazing community forum for Maths teachers. I was a little bit hesitant to give up an opportunity to ‘geek out’ and talk about Maths for a more serious topic but I am so glad that I did. The feedback has been overwhelmingly positive. I have been utterly humbled by the number of Maths teachers who have got in touch to commend the fact that the SMC chose to give a platform to this cause and also to offer their stories, empathies and experiences with mental health and wellbeing. This all goes to show that there has been a definite shift in attitudes towards the topic within teaching – this is greatly encouraging. I love teaching and would love to remain in the profession for as long as possible but as we all know turnover of new teachers is incredibly high. That being said, there are many Maths teachers out there who are struggling and may not be feeling personally the positive aspects and attitudes the profession is cultivating. To those people I would like to extend the same advice I give to pupils: Maths has a vibrant, living, active community of people who are willing to support and challenge each other. Find ways to get connected and make yourself part of the nerd herd! Twitter, YouTube and Maths Jams are a great way to get involved. You are not alone in your love for Maths and you are not alone in your quest to bring the joys of Maths to our amazing young people. The path may be a difficult one to navigate at times but luckily we know graph theory!

References


NHS Scotland, 2019, Module 1: Understanding Health Improvement, lecture notes, Promoting Mental Health Improvement, online course material.


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The relevance of integrating modelling and applications in the teaching and learning of Mathematics is recognized by many teachers and researchers (Barquero, Carreira & Kaiser, 2017). However, there is no consensus about the best way to do it. The main problem is related to the process of adapting a real situation. In many cases the complexity of the mathematics involved requires the introduction of changes in the situation before presenting it to the students. And the result of this process is what Carreira and Baioa (2018) call a loss of the authenticity of the tasks. This means that the adaptation of the task changes it in a way such that the context is still a real one. Nevertheless, it is possible to find in our daily life situations where the mathematics can be addressed at different levels, keeping the situation as it is faced by all of us every day. This is the case with barcodes.

Barcodes are everywhere. They are a mathematical process to represent information and they can be explored to look at different mathematical content at different levels. In this article we address the process of constructing a barcode and we finish by commenting on the mathematics involved and giving some ideas about what can be addressed using barcodes in the classroom.

**The barcodes**

Barcodes are representations of a set of 13 digits, the EAN – European Article Number, which are intended for optical reading (Figures 1 and 2). Its main function is to allow a quick insertion of the digits, while avoiding the risk of mistakes in the insertion process, something that occurs with some frequency when the insertion process adopted is the manual one. Its use is common at the level of trade, at supermarkets and stores, where these codes are used for registration and identification of different items. In this case, along with the bars, the respective digits are displayed. These numbers are then used when problems arise at the level of optical reading.

**Figure 1** Example of the barcode of a Portuguese chocolate

This code gives information on three elements of the product (Figure 3). The first three digits indicate the prefix of the country where the company is registered (560 in the case of Portugal and the chocolate code shown in Figure 1, 761 in the case of the bar code of a Swiss chocolate such as the one in Figure 2, 500 to 509 in the case of a UK item).

**Figure 2** Example of the barcode of a Swiss chocolate

These numbers may give information regarding the country where the product was produced, but not necessarily. For example, a UK company that has a factory in China will have products with a UK code, even though they have actually been produced in another country. Digits 4 to 7 give information about the company responsible for the production. The following digits give information about the product and the last digit is the control digit, which is intended to detect any mistakes in reading the code.

The first digit of this set of 13 digits appears to the left of the bars, and the remaining numbers appear below these, divided into two groups of six digits delimited by tabs (usually longer bars than the rest). The barcode is made up of black or white bars which vary in width. Each digit below the barcode is represented by a set of 7 black or white strips called modules. The various combinations of modules produce different thicknesses of bars. For example, the digit 2 of the second group of six digits in Figure

**Figure 3** The information on a barcode
2 is represented by two black modules, one white module, two black modules and two white modules. During the optical reading the computer assigns to each white module the value zero and to each black module the value one, thus obtaining a binary sequence of length seven (which in the case of our example would be 1101100). Later, the computer associates to this sequence of seven digits a number from 0 to 9, according to the last column of the table in Figure 4.

The digits of the first six-digit set can be represented in two different ways. One of the possible representations corresponds to what we can call the inverse of the representation of the second set of six-digits in which the black bars become white and vice versa. (In Figure 4, compare the first barcode column with the last one.) In this case the digit 2 would be represented by two white modules, one black module, two white modules and two black modules corresponding to the number 0010011. The other possible representation is the mirror image of the coding of the second six-digit group. (In Figure 4, compare the last two barcode columns.) In this case the digit 2 would be represented by two white modules, two black modules, one white module and two black modules, which would correspond to the sequence 0011011. These two representations are designated as even or odd depending on the number of 1s that they include. Thus, 0010011 will be the odd representation of digit 2 because it has an odd number of 1s (three) and 0011011 will be the even representation of digit 2 because it has an even number of 1s (four).

<table>
<thead>
<tr>
<th>Digit</th>
<th>1st digit</th>
<th>1st group of 6-digits</th>
<th>2nd group of 6-digits</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>111111</td>
<td>0001101</td>
<td>0100111</td>
</tr>
<tr>
<td>1</td>
<td>110100</td>
<td>0011001</td>
<td>0110011</td>
</tr>
<tr>
<td>2</td>
<td>110010</td>
<td>0010011</td>
<td>0011011</td>
</tr>
<tr>
<td>3</td>
<td>110001</td>
<td>0111101</td>
<td>0100001</td>
</tr>
<tr>
<td>4</td>
<td>101100</td>
<td>0100011</td>
<td>0011101</td>
</tr>
<tr>
<td>5</td>
<td>100110</td>
<td>0110001</td>
<td>0111001</td>
</tr>
<tr>
<td>6</td>
<td>100011</td>
<td>0101111</td>
<td>0001011</td>
</tr>
<tr>
<td>7</td>
<td>101010</td>
<td>0111011</td>
<td>0010001</td>
</tr>
<tr>
<td>8</td>
<td>101001</td>
<td>0110111</td>
<td>0001001</td>
</tr>
<tr>
<td>9</td>
<td>100101</td>
<td>0001011</td>
<td>0010111</td>
</tr>
<tr>
<td>Parity</td>
<td>odd</td>
<td>even</td>
<td></td>
</tr>
</tbody>
</table>
The choice, for each of the digits of the first six-digit group, between its odd representation or its even representation, determines a six-digit sequence ("1" if the odd representation is chosen and "0" if the even representation is chosen) that allows the first digit of the 13-digit set to be encoded (the one to the left of the barcode).

The table in Figure 4 shows for each digit the encodings: of the first digit of the set of 13, for the digits in the first group of six-digits with odd and even encodings, and for the second group of six-digits. These encodings can be inferred from the analysis of several barcodes or, conversely, can be used to construct barcodes. We will use them to reproduce the barcode of Figure 1.

The first digit of the barcode in Figure 1 is 5, which is coded by 100110. This means that in the first group of six-digits we will choose successively the odd (1), even (0), even (0), odd (1), odd (1) and even (0) representations. This means that for the digit 6 we will choose the odd representation, for 0 the even, for 1 the even, for 0 the odd, for 5 the odd and for 5 the even.

<table>
<thead>
<tr>
<th>6</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>5</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>odd</td>
<td>even</td>
<td>even</td>
<td>odd</td>
<td>odd</td>
<td>even</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>5 (13th digit of the barcode)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The second part of the barcode will be based on the codes of the second set of six-digits.

| 3 | 1 | 3 | 0 | 0 | 5 |

**Barcode control digit**

In an EAN code, the first 12 digits are followed by a control digit. Considering the code as

\[ x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} x_{12} x_{13} \]

where each \( x_i \) represents a digit, the control digit \( x_{13} \) is determined so that

\[ x_1 + 3x_2 + x_3 + 3x_4 + x_5 + 3x_6 + x_7 + 3x_8 + x_9 + 3x_{10} + x_{11} + 3x_{12} + x_{13} \equiv 0 \pmod{10} \]

That is, the thirteenth digit is chosen so that when added to

\[ x_1 + 3x_2 + x_3 + 3x_4 + x_5 + 3x_6 + x_7 + 3x_8 + x_9 + 3x_{10} + x_{11} + 3x_{12} \]

the result is a multiple of 10.

For example, if we want to determine the control digit of code 978972662792?, we would start by calculating \( 9 + 3 \times 7 + 8 + 3 \times 9 + 7 + 3 \times 2 + 6 + 3 \times 6 + 2 + 3 \times 7 + 9 + 3 \times 2 \) which gives 140. The control digit ? would then be a value between 0 and 9 such that 140 + ? is a multiple of 10. That is, 0. Therefore, the complete code would be 9789726627920.

The main function of this control digit is of course to detect any mistakes in the reading of the code. According to Buescu (2002), the most common mistakes are those involving the insertion of a wrong number or the exchange between two adjacent digits. According to Picado (2001), the occurrence of more than one error in the introduction of a single code is very small. It is easy to see that the algorithm for determining the barcode control digit detects cases where a wrong digit is entered. And the exchange between two adjacent figures?

The following table shows the sums of the first two digits in modulo 10 when they are entered in the correct order \( (x_1, x_2) \) or in the wrong order \( (x_2, x_1) \).

An analysis of the table allows us to conclude that the algorithm does not always detect the change in the order of introduction of the two digits. In fact, where the difference between the two digits is five units, the error is not detected. This conclusion can be inferred from the table, but it can also be deduced mathematically. This is due to \( \text{gcd} (2, 10) \) being 2 and...
not 1. Indeed, if we consider the difference between the two control sums (that of the number with error and that of the correct number) we will see that this is twice the difference between the two digits that have been exchanged and which will therefore be a multiple of 10 (and consequently the error will not be detected) where the difference between the two digits is 5.

It thus appears that the control number is not fully effective in detecting mistakes in entering digits, but the truth is that, as Buescu (2002) points out, it is not possible to detect all the changes in the order of introduction of two adjacent digits and all incorrect entries of a digit having as control digit only one of ten possibilities from 0 to 9. However, this was also not the goal. The increasing dissemination of optical reading mechanisms has made digitizing situations much rarer, and consequently fewer processes are required for the detection of their occurrence (Picado, 2001).

We take the opportunity to exploit other control numbers, such as those in bank account numbers (usually two digits and with the intention of reducing the situations in which we transfer money to a different bank account than the one intended), in bank notes and identification documents or passports (find more information on situations like this at www.attractor.pt/mat/alg_controlo/sist_mod_texto_en.html).

In general, whenever we are faced with situations in which multi-digit numbers are involved, that make the occurrence of typing errors natural, Mathematics is also present, giving its contribution to try to detect errors and thus minimize the occurrence of unpleasant situations. After all, nobody wants to deposit money into the bank account of a stranger; just because he was mistaken in the IBAN, nor pay double for a product, due to a mistake of the employee when entering the code of the article. It is to minimize situations like these that we all count on the Mathematics hidden in the various control numbers.

The Mathematics of barcodes

Barcodes can be something to which students may never have paid much attention, however they are used to seeing them and so, they are part of their reality. Some students may find it interesting to learn how they are created and to find an example of a concrete situation in which the utility of Mathematics is visible. Moreover, this is a simple situation, the comprehension of which is possible to students even at the most elementary levels, but which can then be extended and deepened, making it relevant also for students at higher levels. Modular arithmetic, used to define the control digit, is, moreover, one of the areas of Mathematics whose study authors such as Justesen and Hoholdt (2004) consider only to be within the reach of students with some maturity, as is the case of those in 3rd or 4th year at university. The approach may also involve mathematical knowledge at the level of finite field theory relevant to code theory.

At a more elementary level, this may be a good opportunity to provide students some contact with different representations of the same entity, addressing a numerical representation, a geometric representation and a binary representation, which is obviously a numerical representation but different from the one which the students are used to and also allows us to pose them several questions and to establish bridges with areas such as computer science.

It may also be a good opportunity to think more deeply about the numbers and to get to know them better by reflecting on the reason for choosing to multiply half of the code numbers by 3 when determining the control digit. Multiples and divisors will certainly be at the centre of the analysis.

The concept of symmetry can also be approached from the way the two sets of six-digits are formed. Or the relationship between the thirteen digits and the barcode can be approached as a function. But barcodes can also be the starting point for new challenges. If the increase in the number of products leads to the need to pass the code from 13 to 14 digits, how can the barcode be reformulated? Is it possible to consider two digits outside the barcode? And in that case, how can this information be encoded in the bars?

Barcodes are everywhere and we come across them every day without noticing the Mathematics they contain. Building a barcode or perceiving the information it contains helps us to become aware of how Mathematics can be and is useful in our everyday lives. But it can do more than that. It can help to awaken our curiosity and change the way we begin to look around us. It can promote the discovery of the Mathematics that surrounds us. And it can also get our students to begin to use Mathematics effectively and, more than that, to do it in a creative way.

References


The student's sense of credibility.


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I walked into a colleague’s classroom some time back and saw this above the board:

Mistakes are to be expected, respected, inspected and corrected.

Brilliant! This, in a nutshell, was what we had been trying to teach our children for years – that it is not just OK to make mistakes, but mistakes are actually where the learning happens. So don’t rub them out! Instead, value them and talk about them.

I shared this slogan with a Primary 4 class. The children quickly decided that there was a word missing: detected! We had to detect our mistakes before we could respect and inspect them.

With the help of a thesaurus, I found a synonym for ‘Mistakes’ that began with C: ‘Confusion’. We now had:

Confusion is Expected, Detected, Respected, Inspected and Corrected.

CEDRIC

Our Maths working group decided that this was the one thing we should focus on sharing with our children this session. It linked with all the work we were doing on mindset and, if we could embed the message across the whole school, we reckoned it could be quite transformative.

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We launched a competition for our Primary 6 pupils to design CEDRIC. Shortlisted entries were voted on by our Primary 4s and 5s. The winning design (© 2019, George Watson’s College, Edinburgh), produced with a little help from a graphic design artist, is shown at the top of the page.

So how does CEDRIC work in the classroom? In any real learning situation, **Confusion is Expected**. If our brain is being challenged we will sometimes feel confused and we may make mistakes. In order to learn, we need help from others so our confusion can be **Detected**. Peer assessment is great for this. If we and others have different answers, there is an opportunity to investigate! With some of our classes we have introduced the 333 approach, where pupils work in threes, and compare answers after every three questions using a three-step process: ‘plan together, work apart, check together’. Most pupils master this part of the process quickly. A few impatient ones need reminded about team working!

The important next step is that our confusion or mistakes should be **Respected**. We shouldn’t be embarrassed by our mistakes, or hide them, or rub them out. Instead, we should get out our favourite colour and put a bright coloured cross next to each mistake. The children find this more difficult. Some are used to rubbing out mistakes and changing...
them, so retraining has been necessary. The key is to continually remind pupils that the work they do on their own is done in pencil and anything they do with their team is in colour, giving them a clear record of what has been done unaided and how they have improved their understanding with others — great for personal reflection and for sharing with parents and carers.

Having highlighted a mistake, it then needs to be **Inspected**. At this point we encourage pupils to get together with others in their team and talk, in depth, about the differences in their understanding. This is probably the most challenging step in the process; it requires a real change of gear from some children. The objective is no longer to get through lots of questions but instead to engage deeply with the thinking of each member of the team. We encourage our children to use pictures and diagrams to explain their thinking to each other and to write things down. Individual whiteboards are great for this.

We are also finding ourselves increasingly giving support with spoken mathematical vocabulary and sentences. Talking maths is a really valuable skill, but like any skill it requires much practice. We aim to encourage the children to explore their ideas from different angles and agree a solution, confident in the knowledge that each group member can explain everyone’s ideas clearly to someone else. It is early days but we are hopeful that, with persistence, we can bring about real change.

Finally, when the confusion is sorted, mistakes can be **Corrected**. But even now, we don’t rub out the earlier mistake! Instead, we keep it as evidence of our learning and write the correct answer next to it, again using a bright stand-out colour.

Result: if we use CEDRIC well, we have truly learned from our confusion and have jotters or learning logs where everything that we have learned and clarified is clearly highlighted — clear evidence of thinking and valuable information for sharing at Parents Meetings. A teacher’s dream!

### Notes

1. The idea of CEDRIC is not copyright so feel free to use the CEDRIC slogan in your own school. However, the CEDRIC image shown is ours. If you want a CEDRIC image to use in your school, why not run your own competition?
2. See bit.ly/CEDRICMindset for more info and a classroom poster.

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**Notable Mathematics and Numeracy Dates 2020**

Every day is a great day to enjoy mathematics, but we have collated some of the special ones. Check the SMC website for details of upcoming SMC events www.scottishmathematicalcouncil.org/wp1/

- **13th January** UKMT Intermediate Mathematical Challenge deadline for applications www.ukmt.org.uk/
- **24th January** Mathématiques Sans Frontières Scotland deadline for applications
- **7th February** NSPCC Number day
- **1st March** World maths day
- **7th March** SMC Annual Conference in Stirling
- **14th March** Pi day
- **27th March** UKMT Junior mathematical challenge deadline for applications
- **8th July** Math 2.0 Day
- **13th May** National Numeracy day www.numeracyday.com/
- **August** Mathematical Challenge information to schools http://www.wpr3.co.uk/MC/
- **September TBC** MA Stirling Conference
- **28th September onwards** Maths Week Scotland www.mathsweek.scot
- **5th October** World Teachers Day
- **October TBC** UKMT Mathematical Olympiad for Girls deadline for applications
- **23rd October** Mole Day 6:02 am – 6:02pm
- **November TBC** UKMT Senior mathematical challenge deadline for applications
- **November TBC** Enterprising Mathematics in Scotland Final
- **7th November** SMC Northern Conference
- **23rd November** Fibonacci day

*Pythagorean triple day (it happens in 2020 and then the next one is in 2025, try and work out when it is!)*

If you want more, go to www.maa.org/news/on-this-day and check out the Mathematical Association of America’s ‘On This Day’ page. Got a notable mathematics date to celebrate? Tweet us @scottishmaths

*Craig Lowther*
SQA is developing a new, distinctive Higher in Applications of Mathematics that is intended to engage a wider cohort of learners than the current Higher in Mathematics.

The Scottish Government is committed to increasing participation and achievement in science, technology, engineering and mathematics (STEM), to meet the needs of employers and realise Scotland's economic potential. STEM employment is developing more rapidly than in any other part of Scotland's economy and increasing participation in the study of mathematics will help to realise the government's ambitions. Whilst Scotland's participation in Higher Mathematics (approximately 50%) is the envy of the rest of the UK, it is low compared with other parts of the world, e.g. Pacific Rim countries (Hodgen et al., 2010). There is an increasing need for more students to study more mathematics prior to embarking on further study or employment.

Over 70% of all degree courses require mathematical literacy, e.g. data handling and proportional reasoning, which are the skills assessed in the Programme for International Student Assessment (PISA) tests at age 15 (OECD, 2019). Mathematics is like music or sport, where skills atrophy if they aren't used. Studying mathematics for longer means that people are more likely to be able to use these skills in further study, employment and everyday life (ACME, 2011).

Higher Mathematics is a well-respected qualification that is valued by end-users: employers, colleges and universities. However, its emphasis on algebraic techniques, calculus and trigonometry means it is not suitable for all learners, and relatively few candidates with grades B and C at N5 progress successfully.

In England, a number of qualifications have been investigated with a view to increasing and widening senior secondary participation in mathematics, e.g. AS and A level Use of Mathematics and, more recently, Core Maths. Core Maths is designed to engage those learners for whom A Level mathematics is unsuitable. It enables them to maintain and develop their mathematical skills and apply them in a range of contexts that require quantitative and statistical reasoning to solve problems. Core Maths builds on the Use of Mathematics qualifications, using technological tools and a wide variety of authentic real-life contexts to build confidence in using and applying mathematics. Students develop the skills needed for life, work and further study, as identified in ACME's reports on Mathematical Needs (2011) and post-16 Mathematics (2012). The independent evaluation (EMP) of the QCA pilot of A level Use of Mathematics (Noyes et al., 2011) found that the qualification attracted a new cohort of students who previously would not have considered studying mathematics and that numbers studying A level mathematics in the pilot centres also increased.

Scotland is in a position to learn from England's experience and develop a world-class qualification that equips learners with the mathematical skills and confidence needed to prosper in modern society. Following a survey of teachers and lecturers in schools, colleges and universities during summer 2018, the case for the new Higher gained approval from SQA's Qualifications Committee in February 2019 and work is now underway to develop the new National Course. With a view to first teaching in 2021 and first awards in 2022, all materials will be available in summer 2020, to allow sufficient time for centres and teachers to prepare. There is already considerable support for and interest in the new Higher. This includes developing support to upskill teachers. SQA is engaging widely with stakeholders to raise awareness and build the appetite and demand for the new Higher.

This SCQF level 6 National Course will be completely distinctive from the Higher in Mathematics and provide a progression route from National 5 Applications of Mathematics and National 5 Mathematics. The qualification development team for the new Higher has begun meeting and the initial drafts of the rationale, purpose and aims have been prepared. They were shared at the SMC conference and are reproduced here. They are liable to change, but form the basis for developing the knowledge, skills and understanding to be covered, and the assessment design.

**Draft rationale for the Higher in Applications of Mathematics**

The application of mathematics considers a problem from a real-life context, identifies the relevant information, formulates the problem in appropriate mathematical or statistical terms, applies tools correctly, finds a solution, and interprets the solution in the context of the problem.
The Higher in Applications of Mathematics course focuses on the development of the mathematical and analytical skills required in a modern society and for the future workforce. This enables learners to further develop quantitative and mathematical literacy, problem solving and reasoning skills, and to apply mathematics in a variety of real-life contexts, some of which may be complex and unfamiliar. The skills, knowledge and understanding in the course support a wide range of curricular areas including humanities, social sciences, health care, and business.

**Draft purpose for the Higher in Applications of Mathematics**

Higher Applications of Mathematics equips learners with the skills needed to interpret, analyse, and critically appraise statistical and mathematical information; simplify and solve problems; assess risk and make informed decisions by enhancing critical and logical thinking.

**Draft aims for the Higher in Applications of Mathematics**

- select, apply, combine and adapt mathematical and statistical literacy skills needed for life, work and further study in a wide range of curricular areas
- further develop financial literacy in real-life contexts
- use mathematical reasoning skills to generalise, build arguments, draw logical conclusions, assess risk, and make informed decisions in familiar and unfamiliar situations
- use a range of mathematical skills to analyse, interpret, and present data and numerical information
- appraise quantitative information critically in the light of modelling or statistical assumptions
- use appropriate digital technology to manipulate and model mathematical, statistical and financial information

SQA is keen to receive your impressions, views and hopes for this new course. Please e-mail martin.brown@sqa.org.uk.

**References**


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Introduction

I gained a Masters in Maths Education about four years ago. From this I became interested in the CPA approach to teaching maths and carried out further research, sharing practice with colleagues, parents and teachers.

For the past two and a half years I have worked as a Pupil Equity Funding (PEF) Officer, Raising Attainment in Maths, with a focus on developing the CPA approach in Victoria Primary School and the Graeme Cluster.

Where to start?

Upon arriving at the school I found that children had little prior knowledge of the connections between maths concepts. Their basic understanding of maths, particularly number, was limited and retention was poor. They relied heavily on the 'bus stop method', 'chimney sums' or 'tricks' for simple calculations. A lack of confidence and inability to solve problems in unfamiliar situations was also apparent. This led me to explore where misconceptions were being formed and question why, for example, P7 children were unable to explain the value of the digits in a number, positively about maths.

I began immersing myself in what I understood to be the CPA approach. Over the years, many mathematicians have offered theories on this approach, including Jerome Bruner who developed it in 1960.

\[ \text{CPA approach gives benefit to the most students and has proven to be very effective to help students who have difficulty in learning mathematics, because the CPA approach is moving gradually from actual objects through image and then subsequently to the symbol (Jordan, Miller, & Mercer).} \]

The Education Endowment Foundation (2017) supports the view that concrete manipulatives allow students to make links between the maths concept and the material. It is therefore vital that consideration is given to how concrete materials are used.

The CPA approach is not a linear approach whereby one concept is taught, the children learn it and immediately move onto the next concept; rather it is ‘spiral’. There is no timeframe set against particular topics because the emphasis is on revisiting and consolidating. All pupils, even the more advanced learners, need time to immerse themselves in the concrete and pictorial stages, rather than jumping too quickly to the abstract, in order to reinforce and embed concepts and deepen their understanding.

When introducing concrete materials for the first time, it is important to give pupils the chance to play with and explore the resources to see what connections can be made naturally. This allows the teacher to observe and discuss the children’s ideas and note any misconceptions revealed.

In January 2019 I was fortunate to visit a school in Leeds to see a teacher from Shanghai use this approach first hand. There was a focus on embedding key themes, applying variation and moving subtly between the concrete, pictorial and abstract stages, making links with real life. It drew my attention to the fact that sometimes we rush through content, rather than simplifying ideas. Further information about this project can be found on the NCETM website.

Developing CPA in school

In school we plan using free material from the White Rose Maths Hub, which incorporates the Mastery approach, whilst making links to Falkirk Council guidelines and the National Benchmarks.

We place a strong emphasis on number. In the early stages, tens frames are used to scaffold the children's thinking and build sound number sense. We construct sentences such as, 'There are three blue counters and five yellow counters so there are eight counters altogether.' Encouraged by Bernie Westacott, we ensure that formal addition is not introduced too early. We use a number of other concrete materials (for example, beads, loose parts and coins) and encourage the children to play with these (Figure 1). During all aspects, the children are encouraged to talk about what they see and formulate their own ideas.

We also use Numicon alongside five/tens frames, for example for number bonds, doubling and partitioning. Children use weighted Numicon pieces and a two-pan balance to investigate statements such as \( 8 + 2 = 5 + 5 \) (Figure 2).
After the children have grasped an understanding of number with everyday objects, tens frames and Numicon, I introduce them to Base 10 blocks. We had lots of these materials gathering dust in a cupboard!

Base 10 materials can be helpful for teaching numerical value and reinforcing place value. Vocabulary like ‘exchanging’ and ‘regrouping’ is introduced and used when discussing the abstract representation. Base 10 blocks can also be used to aid understanding of the decimal system (Figure 3).

Place value counters are another useful tool for developing children’s understanding of Base 10. As with Base 10 blocks, we use place value counters to illustrate that 10 lots of 0·1 equals 1, not 0·10 (Figure 4). This is useful for showing how we do not just ‘add a 0’ when multiplying by 10.

As children become more familiar with concrete manipulatives, I hope that they will become more confident about choosing the most appropriate resource to use to help them understand.

Pictorial images I have used are the bar model and part-part-whole (number bond) diagram. The children use these images for addition and subtraction and to develop their understanding of equal parts/quantities (multiplication, division and fractions). The language of ‘parts’ and ‘whole’ is important in helping children make connections and explore mathematical relationships, for example, 6 tenths and 9 tenths = 15 tenths or 1 one and 5 tenths. It is worth noting that the orientation of the bar model should be varied, as should the position representing the whole with the top bar.

Summary

As I supported children and teachers further up the school, I became increasingly aware that sometimes we move too quickly to the abstract which can result in misconceptions. This further highlighted the need to teach concepts differently and strengthened my commitment to the CPA approach.

Many people often ask where I get my ideas and resources from. There are a number of people who
Taking the First Steps Towards Teaching for Mastery

Gillian Mathewson

With many of our cluster primary schools implementing a ‘Singapore’ Maths mastery programme, and the local authority running mastery twilight sessions for primary teachers, I felt the need to find out more about maths mastery. We were going to have a group of pupils who were taught maths in a particular way, and another group who were not, and I wanted to understand this way of teaching, and see if there was anything I could learn from it to take into my own practice.

I initially struggled to find resources relating to secondary teaching and was finding it difficult to work out how to take the lesson by lesson scripted primary resource and apply it to secondary teaching, without it being a mountain of work. In researching mastery online I found myself embedded in conflicting views and information, and I was struggling to work out where to start. The 2018 SMC conference changed all that. Three presentations had a massive impact on my understanding: Stuart Welsh’s keynote; Fiona Phillips presentation on using concrete materials in the secondary classroom; and Chris McGrane’s presentation on teaching for Mastery.

There are 4 key changes that I have made to my practice as a result of the conference, and then further reading, presentations and conferences:

- When appropriate I use concrete materials and/or representational diagrams. This will depend on the topic and the resources available, but I will not try to force concrete and pictorial representations into everything. In the future I expect that I will add to my repertoire of topics that I teach in this way, as I learn about different resources, and share ideas with colleagues.

- When we are working in the abstract representation of maths, I will try and find/produce tasks that introduce concepts step by step. This is based upon reading and

References


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presentations surrounding Variation Theory, Differentiation and Cognitive Load Theory.

• Once initial concepts are understood, I look to extend into mathematical thinking. This is not just as extension work, but for all pupils, sometimes as a starter task, sometimes at different stages through a differentiated task.

• Low Stakes Assessment. In the Broad General Education (BGE) stage, I give my classes a weekly 'checkup' low stakes assessment. This is marked by peers in class, and gives me, and pupils a picture of where their strengths lie, and what they need to work on.

Concrete / Representational Approach

When I went home from the 2017 conference, I made myself a set of algebra tiles using Foam and PVA glue, and I used these to demonstrate the concept to colleagues who had not attended the conference.

Having just finished teaching factorising quadratics, we then used these to support pupils who had struggled with the topic, and while some pupils did not like them, they had a positive impact with others. As a result of this we were able to demonstrate the benefit of these resources to our Senior Leadership Team, who purchased class sets for the entire department. To support my colleagues I created a series of lessons for our introductory algebra course for S1 using the algebra tiles, which was available for all to use. This was quite a challenging task, as having never actually taught the content using the algebra tiles, I was creating a starting point, and the response from the pupils was very positive. I will be really interested to see the development in algebra from this group of pupils as they progress through the school.

I am continuing to explore different ways to use these resources and to think about how to teach using a concrete – representational – abstract approach, as well as exploring different concrete resources. This has been further developed through attendance at the ATM Branch meetings in Glasgow and sharing ideas with colleagues on Twitter.

In planning to teach volume to my 3rd year class, I was seeking a concrete resource to reinforce the concept of volume and found a bucket of 1000 multi-coloured plastic cm cubes on Amazon. These proved to be a great way of helping them to understand the 3D nature of volume and through being able to see the difference between, say, 12 cm, 12 cm$^2$ and 12 cm$^3$, the importance of units. As a concrete resource these have been very adaptable, and I have found myself using them in teaching fractions and ratio, as well as for probability and statistics.

I have also been experimenting with Cuisenaire rods. I am still in the early days of my understanding of how to use them, and with only 1 set will be limited to which classes I can use them with, but we have been exploring their use in linear patterns and symmetry.

My final go-to resource is paper. I am a big fan of origami and will often use origami in a last lesson of term task. I have a range of origami paper, and when teaching similarity and scale factor with my S3s, I found myself looking to see if I could demonstrate area and volume scale factor through folding.
I attended the ATM branch meeting in Glasgow where Mike Ollerton was speaking, and he demonstrated an excellent task with fractions which got a whole room of teachers engaged and keen to see what happened next, and I found it had the same effect on my pupils. The task involves folding triangles and cutting them, resulting in a range of equivalent fractions making up a variety of shapes. I can see the potential for this task to be used in a number of other topics, such as area and similarity.

**Variation Theory / Cognitive Load**

Once I have moved into the abstract phase of learning, one of the major changes I have made in my practice involves careful consideration of the tasks I give to pupils. I realised that often I was giving pupils tasks where there were two new concepts at once, and they were struggling to cope. The tasks I gave pupils were fundamentally the same question repeated with different numbers. Chris McGrane’s presentation at the SMC 2018 conference made me think deeply about the questions I give pupils, and this was reinforced through attending the ATM Branch meeting with Anne Watson and John Mason presenting. Anne and John spoke about how you can draw attention to aspects of maths through looking at what changes, and what stays the same. For example, the task below was used in the teaching of multiplication and division of integers. Through including only the multiplication of 4 and 6 to give 24, and then combinations thereof, pupils’ attention was drawn only to the aspect of maths I was looking for them to understand, so reducing cognitive load.

<table>
<thead>
<tr>
<th>4 x 6 =</th>
<th>(-4) x 6 =</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 x 4 =</td>
<td>6 x (-4) =</td>
</tr>
<tr>
<td>24 + 4 =</td>
<td>(-24) + (-4) =</td>
</tr>
<tr>
<td>24 + 6 =</td>
<td>24 + 6 =</td>
</tr>
</tbody>
</table>

I have also observed with my classes that because the variation offers the opportunity to pattern spot, and verify their answers, it has increased their confidence. Pupils are more likely to show correct working because as they already know what the answer is, they are verifying that it can be reached through what is being taught.

I have found with this method that there is also opportunity to get pupils to create their own questions which will fit into the variation sequence. The topic of Surds offers a great opportunity for this, as the sequence is less evident than in many other topics (see top of the next page).

**Thinking Mathematically**

I saw Gary Lamb speak at a Complete Mathematics conference in Edinburgh, where he was discussing ways in which he offered extension to his pupils. He would give them an exercise with 5 or 6 questions in it. He would then extend pupils with a backwards-working type question, then finally ask them to make up questions of their own: one straightforward and one more challenging. I really liked this, and have used and developed it with my classes, asking pupils what about the question makes it straightforward and what about it makes
it challenging? I then ask them to take a question and make it more challenging in different ways. As a result of this I have had complexity added through the use of fractions, negative numbers, and algebra. The pupils are seeing the interconnections between what they are being taught and are challenging and extending themselves.

I have been inspired to develop my own mathematical thinking through occasionally logging in to the ATM’s fortnightly #BeingMathematical on Twitter.

**Low-stakes Assessment**

I have been giving my BGE classes a weekly low-stakes assessment, designed to give them and me a picture of their understanding of the topic, and what they need to work on. While this has worked quite well, in particular with pupils taking their marked assessment away to correct any mistakes, it has added significantly to my workload, both in terms of creating the assessments and recording and reviewing results, so I am currently thinking about how to amend how I do this to make it more efficient.

**Impact and taking the first steps**

The impact on my classes has been quite significant. In one class in particular, I have seen a significant improvement in engagement and a reduction in low-level disruptive behaviour, attainment has increased, and pupils seem to care more about their learning. The feedback I have received from pupils has been very positive; they like the way their work is being structured and will tell me if I am going too fast or covering too many steps at once.

With this years’ S1s, many of whom had come through P7 doing a Singapore Maths scheme, I have found them very receptive to this way of working, although they have challenged me to find new ways to extend them, as they have seen many of the publicly available resources such as White Rose Maths before. They were also not very keen on bar modelling, as they felt that every question was being forced into a bar model and were more appreciative of being able to use a variety of different models.

**Tips for getting started:**

- Don’t try and change everything overnight
- Choose a topic and think about what you can change – then give it a go!
- Your pupils will need time to adjust too!
- If you’re not on twitter – join. It’s a great place for sharing ideas
- Look at resourceaholic.com and https://startingpointsmaths.blogspot.com/ for great resources
- Find a learning ‘buddy’
- Go to Conferences / CPDs / ATM Branch Meetings.

I have been asked a number of times when speaking to colleagues about the work I am doing, whether we as a school are following a mastery curriculum. We are not. I have gone on a ‘fact finding’ mission with the blessing of my Principal Teacher, and our Senior Leadership Team, but I am working within the realms of our existing schemes of work; so have only been implementing change at classroom level, and discussing and sharing ideas with colleagues. I still feel like I am only scratching the surface and I know I still have a lot to learn.

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Most children love K’NEX. Unfortunately, some don’t love maths. My experience suggests that K’NEX can be used as a tool to reach out to and connect with these reluctant mathematicians, sometimes without them even realising… until later.

You might not know it, but you were developing your maths skills in a fun way.

P6 pupil

Many schools already have K’NEX which is great as it means there is no expense involved in starting to use it for maths. Hopefully, after reading this article you will sort out your K’NEX and start exploring how it can be used to support your pupils on their journeys to mathematical enjoyment and success. There are four different ways to get started:

1. Use the K’NEX you already have and request a copy of the free ‘K’NEX for hands-on Maths’ Guide.

Option 1

The ‘K’NEX for hands-on Maths Guide’ is available from both K’NEXT Generation and the K’NEX User Group. However, the K’NEXT Generation version has been revised to align with the Scottish Curriculum for Excellence (CfE). The Guide contains 101 different K’NEX activities that can be used to enhance and support maths teaching and learning. The activities are not just about building with K’NEX, they are about using K’NEX in a way that deepens mathematical understanding and enjoyment.

The activities in the guide range in complexity from sorting and counting, creating numerals, exploring 2D-shapes and number lines, to building analogue and digital clocks, theodolites and measuring wheels.

All activities are illustrated using Classic K’NEX but many can be adapted for use with Kid K’NEX at Early and First Levels. Using Kid K’NEX in this way allows for a seamless progression to using Classic K’NEX. The more complex activities are suitable for Second and Third/ Fourth Level pupils, including more able or talented children. The activities are also suitable for children with additional support needs.


3. Arrange a K’NEXT Generation pupil workshop in your school.

4. Follow K’NEXT Generation on Twitter.
It was maths but it didn’t look like maths and it was fun doing it that way.

P5 pupil

Also included in the Guide are ideas for costing exercises, researching ratio, exploring 3D-shapes, practising co-ordinates, appreciating 2-D and 3-D symmetry, understanding nets, and carrying out maths investigations which include data collection and graph construction.

Option 2

If you would like a little more support to use the guide (and a chance to play!), K’NEXT Generation can visit your school and deliver a ‘just for teachers’ workshop. During the workshop you will get a chance to try out some of the activities in the Guide with the support of Scotland’s K’NEX Training Consultant. This workshop also provides a great opportunity to share, discuss and develop ideas with colleagues in a fun, relaxed and supportive atmosphere. A free electronic copy of the Guide is included.

Excellent, gave me lots of ideas as a teacher of ways to make problem solving fun.

P5 teacher, Aberdeenshire

A great way to use child friendly materials to engage students with maths learning of all types.

P5 teacher, Aberdeen City

Very relevant for exploring understanding across maths concepts.

P5 teacher, North Lanarkshire

Option 3

K’NEXT Generation also offers a K’NEX Maths Workshop, designed for P5-7 pupils. This workshop uses some of the activities from the Guide along with other maths puzzles that have been adapted for K’NEX. The workshop has been designed to challenge and engage pupils and encourage teamwork and problem-solving skills. While travelling around the world with a partner (without leaving the room!) pupils collect stamps in a mini passport by solving mathematical puzzles.

It was extremely fun and a better and different way of doing maths.

P5 pupil

It was so creative and so cool doing maths with K’NEX.

P6 Pupil
Maths isn’t something you just write on paper.

P5 Pupil

An active and exciting way to use and develop problem solving skills & strategies.

P5-7 teacher, Highland

Option 4

Follow K’NEXT Generation on Twitter to see a series of tweets about K’NEX Maths Challenges that you can try out with your class. All challenges use basic K’NEX pieces that are already in many schools. You and your pupils can join in by tweeting photographs of your solutions to the challenges or tweeting your own challenges for others to try.

If you decide you would like to explore the benefits of using K’NEX in maths, then hopefully the above ideas, photos and comments from pupils and teachers will encourage you to get started. Meanwhile, I will leave the final word to one P7 pupil.

K’NEX makes maths fun!

Notes and useful weblinks

- K’NEXT Generation is the K’NEX Training Consultancy for Scotland, delivering CfE-linked K’NEX workshops for both staff and pupils throughout Scotland.

- The K’NEX User Group is a not-for-profit organisation offering a wide range of free resources. The group also hosts Europe’s largest K’NEX shop, which sells K’NEX sets and parts: https://www.knexusergroup.org.uk/

- Information about the K’NEXT Generation ‘Just for teachers’ workshops and pupil workshop: https://www.knextgeneration.co.uk/

- All comments are taken from evaluation forms that were completed by pupils and teachers following their participation in K’NEXT Generation’s pupil or teacher maths workshops.

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Sudoku

In sudoku, each row, each column and each box contains the whole numbers from 1 to 9. Complete the sudoku shown, inserting each number when you are clear that it is the only possible number for that cell. The solution is given on page 77.

Chris Pritchard
Advanced Collaborative Problem-Solving in Mathematics

Paul Argyle McDonald

Introduction
There is a growing body of literature that recognises the importance of collaboration as a fundamental societal skill within an interconnected world. It has been reported that a major challenge of the twenty-first century is to prepare individuals for participating in organised teams to solve real-life problems. Education systems around the world are tasked with training young people for the workplace (Fiore, Grasser and Greiff, 2018). Within the mathematical framework of Curriculum for Excellence, teachers have a duty to orchestrate opportunities for pupils to experience collaborative problem-solving (e.g. Scottish Government, 2009). In this article, I briefly explore the role of teachers in collaborative problem-solving and present a sample of pupil solutions generated from a Fife primary school.

What is collaborative problem-solving?
Viewed simplistically, this construct may be described as a problem-solving activity involving collaboration between individuals. A broader perspective has been adopted by Hesse et al (2015, p. 39) who argue that collaborative problem-solving is "a joint activity where dyads or small groups execute a number of steps in order to transform a current state into a desired state". This delineation captures some important features but is inadequate since it does not specify the essential cognitive components found in individual problem-solving or collaboration. According to the OCED (2017), collaborative problem-solving is:

- the capacity of an individual to effectively engage in a process whereby two or more agents attempt to solve a problem by sharing the understanding and effort required to come to a solution and pooling their knowledge, skills and efforts to reach that solution (p. 47).

What is appealing about this definition is that it recognises the inherent complexities involved. For example, in order to ‘come to a solution’, interdependency is suggested to be present among team members. However, underpinning this process is an understanding of the nature of mathematical problem-solving. A number of studies have reported that teachers hold misconceptions about the structure of a mathematical problem. For example, McDonald (2017) found that in his study of Scottish primary and mathematics teachers’ beliefs, almost half of the participants considered a mathematical problem is classified by the blending of words and a routine algorithmic task. Though, it is a widely held view that a problem must contain an element of challenge that provokes learners to extend their mathematical thinking (e.g. Mason, Burton and Stacey, 2010).

Facilitating implementation
Tabach and Schwarz (2018) draw a distinction between the role of mathematics teachers during whole class instruction and collaborative group work. The authors assert that facilitating collaborative interactions is multidimensional and complex to manage. Moreover, assigning pupils to groups does not guarantee that learning will occur (Webb, 2009). Instead, teachers have various duties to perform to establish a productive outcome. However, Xenofontos and Kyriakou (2017, p. 144) highlight that many teachers avoid collaborative problem-solving due to “seeing themselves as weak problem-solvers or lacking the pedagogical knowledge of communicating mathematics, as well as not being familiar with collaborative grouping as a pedagogical approach”. From my professional experience over the last few years, two common narratives have emerged from secondary colleagues that act as barriers to collaborative problem-solving. First, the pressures of ‘exam accountability’ suggest that some practitioners elect to concentrate their time on whole-class teaching. Second, individuals have expressed feeling uncomfortable with not being in control of noise levels, emphasising that pupils often manipulate the activity by going off task to socialise with their friends. On the other hand, the literature has also reported that pupils lack collaborative skills. For example, in her study of American primary pupils, Barron (2003) found that during collaborative interactions, children exhibited inappropriate interpersonal and teamwork actions that impeded the functionality of the group and individual learning. Indeed, many readers may relate to the fact that some less-capable children prefer working individually for a variety of reasons, such as feeling marginalised by being unable to contribute as much as others.
Why should teachers promote collaborative problem-solving?

Collaborative problem-solving offers multiple educational benefits to the learning of mathematics. For example, the contribution of the group can be greater than the sum of the outputs from individual members. Verbal communication and logical-reasoning skills are developed during group interactions. Pupils share different problem-solving strategies which can generate alternative solutions. Comparing and discussing multiple approaches to a problem facilitates learning, particularly procedural knowledge and flexibility (Rittle-Johnson and Star, 2007). Furthermore, due to the heterogeneous composition of classes, less-confident pupils can be empowered to learn from more-able pupils in a non-threatening environment. Boaler (2008, p. 192) highlights that low-achieving pupils “typically rush into answering problems without planning systematically, neglecting the use of key strategies, and finishing when they have found an answer without stopping to consider whether the answer was reasonable”. Peer support can help to address this type of mathematical behaviour. In the same vein, Watson and Chick (2001, p. 166) argue that “one group member may lift the others and then from that cognitive platform a different member of the group may lift them still higher”.

In this next section, I present two tasks that I have used with a range of age groups and abilities, the focus of which is to share my experiences of a recent invitation to a primary school in Fife. With kind permission from the headteacher and in agreement with the class teacher, I was granted access to a P7 class. This class consisted of 21 pupils arranged in advance into six comparable groups. In setting out objectives, children were encouraged to respect the opinions of others, share their thinking, mutually agree on a problem-solving strategy, and provide at least one alternative solution. Significantly, the pupils were offered no scaffolding with any of the tasks.

**Problem 1** (Time allocated = 45 minutes)

In a room of ten children, everyone shakes hands with everyone else exactly once. Find the total number of handshakes. [Answer is 45 handshakes]

The groups began this task by shaking hands with each other. However, they quickly realised that they did not have an efficient system to record the increasing number of handshakes. The following sample solutions were generated.

**Solution A**

This elegant solution is based on searching for a pattern. As observed in Figure 1, the right-hand column contains the sequence of triangular numbers. On further inspection, it can be seen that half of the product of the number of children and the number of children on the previous row equals the total number of handshakes. For example, if we select 5 children, then the total number of handshakes in the room is given by \( \frac{1}{2} (5 \times 4) = 10 \).

![Figure 1 Sample solution for Problem 1](image)

**Solution B**

A visual-representation strategy is presented in Figure 2 (next page), which includes points and pathways to represent handshakes. Such a skilled approach is equivalent to using the formula for the sum of the first \( n \) natural numbers, i.e.

\[
\frac{n(n-1)}{2}, \text{ where } n \geq 2.
\]

**Problem 2** (Time allocated = 45 minutes)

In a restaurant there are four seats in a row. In how many ways can four friends be seated together? [Answer is 24 ways]

Throughout this task, pupils adopted a practical approach by using chairs and directing one another to move around. Here are two sample solutions.

**Solution C**

The strategy adopted here is to account for all the possibilities. It can be seen in Figure 3 that the options are listed in a systematic way. Constructing such an exhaustive list allows straightforward examination of all the possibilities. The use of colour enhances the quality of the solution.

**Solution D**

The sophisticated approach (normally associated with high-achieving secondary pupils) illustrated here appears to be a combination of two strategies.
By constructing a table as shown in Figure 4, the solution is derived by first converting the problem into a simpler form, followed by pattern recognition. Interestingly, by revealing their thinking, this group have demonstrated metacognitive awareness.

Metacognition is a key variable during mathematical problem solving. Furthermore, collaborative talk and metacognitive talk have been found to be mutually mediating, suggesting that the process of collaboration may enhance metacognitive talk and vice versa (Smith and Mancy, 2018). Moreover, a great future opportunity exists for the teacher to extend mathematical thinking by exploring factorials. For example, $4! = 1 \times 2 \times 3 \times 4 = 24$.

Antecedents explaining this collaborative experience

Much work had recently been undertaken by the school to promote collaborative problem-solving during mathematics lessons. The class teacher was acutely aware of the challenges faced while structuring collaborative activities such as selecting relevant materials, keeping children on task, establishing an ethos that fosters the exchange of different perspectives about solving problems and monitoring overall individual performance. It was evident that pupils coordinated closely with each other, allowing the emergence of new ideas and the rejection of incorrect explanations. In particular, it appeared that positive interdependence was embedded as a classroom norm.
Discussion

Despite collaborative problem-solving in mathematics offering a plethora of benefits to learners, the extent to which it is operationalised in primary and secondary schools in Scotland is unknown. Accepting that there is a wide range of functions to be performed to achieve a productive outcome, teachers are obliged to acquire an assortment of pedagogical skills. However, while challenges associated with implementing the collaborative process may overwhelm some individuals, it is important to note that these can be overcome with careful planning and practice, especially if these skills are rooted in the whole-school experience from an early stage. The growth of collaborative skills is not limited by age. For example, Palmer and Van Bommel (2018) found in their study of Swedish primary pupils that children as young as six are capable of working together to solve mathematical problems without any negative effects. Crucially, there has to be a willingness to relinquish a degree of classroom control for the advancement of pupil learning. Although given that collaborative problem-solving is not embedded within any assessment framework of the Curriculum for Excellence, it might be impossible to convince all practitioners of its merits. Further research is required to explore the barriers and facilitators to teachers employing collaborative problem-solving pedagogies. Evaluative research would also develop a deeper understanding of the relationships between classroom approaches and successful collaborative problem-solving.

References


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Despite the best efforts of teachers, mathematics can be seen by teenagers as a dry, boring and mechanic subject, involving no creativity. Many people also tend to think of Mathematics as a synonym of numeracy, ignoring many areas within the discipline, and many sectors where Mathematics play a prominent role. How can we show that there is much more to mathematics than numbers? How can we help our teenagers to develop an appreciation of Mathematics and show them the range of careers this discipline can open up?

*Maths Week at Work* is a series of videos for secondary school students, produced in September 2018 by the University of Edinburgh, with support from the Scottish Government (see the end of the article for details). The series consists of five episodes. In each episode a Maths graduate explains how they currently use Mathematics in their jobs. They also talk about the struggles they faced while studying and how they overcame such obstacles. Finally they present and solve a puzzle which is somehow related to their job.

**Main aims of the series**

This project has multiple but related aims. First of all, we aimed at giving an idea of the breadth and applicability of Mathematics, as well as the wide range of careers this discipline can open up.

Particular attention was paid to presenting some non-numeric areas of Mathematics which are not necessarily part of the school curriculum. For this reason some of the puzzles focus on areas such as Geometry, Topology or Logic and Problem Solving. Some of them are meant to create a sense of surprise, by introducing mathematical concepts that people do not naturally associate with Maths.

We tried to give an insight into Mathematical thinking, which involves conjecturing, trial and error, learning through mistakes, and often getting stuck. This mental process is shown through the characters’ attempts at solving the puzzles, their struggles, their mistakes, their unexpected progress and their satisfaction when they manage to complete a task which initially appears impossible.

In order to defeat Maths anxiety, we tried to pass on the idea that even people who like Maths sometimes struggle with it, and that making mistakes is an important part of the learning process. For this reason we asked each of the graduates to talk about the struggles they met while studying Maths, and how they overcame such struggles.

Furthermore, some of the dialogues are aimed at raising awareness of the creativity behind Mathematics, and the fact that Maths is not a static subject, where everything has been discovered a long time ago, but rather a dynamic and active discipline where new theorems and formulae have recently been discovered, or are yet to be discovered.

**The graduates and the puzzles**

The Maths graduates who took part in the project work in a range of sectors, from the videogame industry to the actuarial sector, from education to the transport industry. The person interviewing them, called Hannah, genuinely dislikes Maths at the beginning, but the conversations with the graduates prompt a positive change in her attitude towards the discipline.

An overview of the whole series and the puzzles is given below.

**Day 1: Geometry and videogame design**

The main character of this episode is Dr Richard Archibald. After a degree in Astrophysics and a PhD in Mathematics, Richard went on to work as a videogame programmer. Many areas of Mathematics, e.g. Geometry, play an important role in his everyday job. The puzzle is an example of the kind of thinking one has to do in game programming every day.

**The puzzle**

Consider the floor plan of an art gallery. Suppose that we know the room has a polygonal shape (the perimeter is only made of straight lines). We would like to place some cameras to be able to monitor the entire room.

- Cameras can only be placed in the corners
- Cameras can see all around, but obviously they cannot see through walls.

What is the lowest number of cameras we need to use if we want to monitor the entire room?

Of course the answer will depend on the shape of the room. Try to solve the puzzle with the following shapes:
Surprisingly, whatever the shape of the room, we can always monitor the entire room, using a number of cameras which is no bigger than the number of corners divided by 3. If you are curious to know why, watch the solution video.

**Day 2: Maths in the insurance sector – how can Maths help a sports star?**

The main character of this video is Caitlin Stronach, who after getting a degree in Mathematics, started working as an actuary for an insurance company. Rather than a puzzle, Caitlin poses an open-ended question, prompting us to think of an actuarial problem in the context of sport insurance, and showing an example of how Statistics can be used in everyday life.

**The puzzle**

A sports star wants to find out how much it would cost to insure themselves against the unlikely event that they would not be able to play sport anymore. If you are the person who is selling the insurance, what kind of factors would be most important to take into consideration?

**Day 3: Maths, patterns and problem solving**

The main character is Dr Steven O’Hagan. After pursuing a PhD in Mathematics, and working as a secondary school teacher, Steven became deputy director of the UK Mathematics Trust. The puzzle is aimed at developing mathematical thinking. As Steven mentions, the everyday job of a mathematician involves studying the underlying structure of patterns, finding an optimal way to reach a solution and constructing a logical argument.

**The puzzle**

We have sixteen counters which are black on one side and white on the other. They are arranged in a 4-by-4 square. Initially, all the counters are facing black side up.

In one move, we must choose a 2-by-2 square within the square and turn all four counters over. For example, we could turn all four counters in this square.

Describe a sequence of moves of minimum length that turns the initial configuration into one where the counters alternate in colour, as shown below.

How can you be sure this is really the minimal number of moves you need?

**Day 4: Maths and transport planning**

The main character of this video is Tessa Hayman, who studied Maths and is now working as a transport modeller. The puzzle she presents is a transport problem she has to tackle in her job. The puzzle also gives an example of how a complicated question can be approached in a systematic way.

**The puzzle**

We have a road network with four exits: A, B, C and D.

A survey company has provided junction counts for each of the three junctions. The junction counts tell us how many cars make each turn in one hour (see pictures below).

However, we do not know how many cars go from exit A to exit B, since this would be very expensive to survey. Can you work out, from the junction counts, how many cars go (in one hour) from exit A to exit B?
Day 5: Maths, knots and tangles

I am the main character here. I work as the Maths Outreach Coordinator at the University of Edinburgh and the puzzle comes from Topology, the area of mathematics I focused on during my doctoral research. This area is concerned with the study of shapes. The puzzle is aimed at generating a sense of surprise (by showing an area of Maths many are unaware of) and at developing basic skills in topology.

The puzzle

We have a necklace, and a magnetic board with two supports: a white support on the left, and a blue support on the right. Hang the necklace in such a way that all three of the following are true simultaneously:

1) If both supports are there, the necklace will hang
2) If we remove the blue support the necklace will fall
3) If we remove the white support the necklace will fall.

HINT: The solution is not related to balancing or creating friction. You should imagine the chain is perfectly smooth and there is no friction. The trick only relies on how you hook the chain around the two supports. The chain would be tangled if both supports are there, but, as soon as you remove either one of the two supports, it will untangle.

Solutions

Solutions to the above puzzles can be found on the project webpage. As well as giving the solution to the puzzle, the second video of each episode contains useful insights into Mathematical thinking.

How to use the videos in the classroom

Even though the video series was initially produced for Maths Week Scotland, it can be used any time of the year, and over several weeks. The videos are in principle aimed at S1-S4 students; however, they might be suitable for younger or older students as well. Though challenging, the puzzles do not require much background knowledge.

As for how to use the series in the classroom, that will depend on context. We would advise that you show no more than one episode per day. We recommend you show the first video of the episode, so that the children get to hear about the different graduates, their jobs and their struggles; then you give your students some time to work on the puzzles in groups, and try to find the solution themselves.

We recognise that some of the puzzles are challenging, and we recommend that you give hints if your students are at the point of giving up. The videos themselves give hints. On the project webpage you can also find some “tips for teachers”, with possible tips to give students should they get stuck.

We do not expect all students will be able to get to the solutions, and in this case, even getting a real understanding of the given solution is a big achievement. One of the lessons to learn from the puzzles is that often a very complicated question can be broken down in simpler steps.
After discussing the solution with your students, we would recommend that you show the solution video, since that will give some insights into mathematical thinking. We also reckon it would be useful, after watching an episode or the full series, to have a discussion with the class, based on questions such as:

- Is there a key concept you learnt today? (e.g. mistakes can help me learn)
- Any surprising fact about Maths or mathematicians that you did not expect? (e.g. Maths is useful in videogame development; many areas of Maths do not involve numbers or equations).

You do not need to watch the whole series; you can pick only some of the episodes. However, we would advise that you show both the introduction (beginning of the first video) and the conclusion (end of the last video), to see the change in Hannah’s way of perceiving Maths.

Feedback from teachers

Here is some feedback I have received from teachers who have used the project in the classroom. All of the teachers who gave me feedback have used the project in front of the full class. Most of them used only part of the series. They selected three or four videos out of five based on the graduate’s job and the suitability of the puzzle. They all valued the interviews and puzzles, since those helped putting Mathematics into context.

Below are some selected quotes from teachers:

‘The classes really enjoyed doing something a bit “different” to normal.’

‘The activities really engaged pupils; they enjoyed seeing how maths could be used in real life situations.’

‘Pupils enjoyed seeing the “why” behind learning and it motivated them to try harder.’

‘Problem solving tasks helped build resilience.’

‘Pupils learnt that, although they may not “enjoy” maths, jobs that they are interested in include maths skills that can be learned through hard work.’

‘The puzzles were suitable for many year groups.’

‘The activities didn’t need any overly complicated maths so were accessible to a lot of pupils.’

Some teachers suggested having puzzles that cater for different levels of abilities. To address this issue, I posted some documents online, containing tips on how to help students who get stuck, as well as possible follow-up questions. I will act on this suggestion when planning future projects, and endeavour to choose low-threshold high-ceiling puzzles which are easy to get into, but give the chance to go deeper.

Unfortunately, the data I present above are based on a very limited sample. In case you end up using (or have used) the series in your classroom, it would be really beneficial for us to receive your feedback; that can really help us to improve future projects.

How to find the series

The video series, as well as helpful templates and tips for teachers, can be accessed through the following website:

www.maths.ed.ac.uk/school-of-mathematics/outreach/mathsweekscotland/maths-week-at-work

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SMC Bursaries for Teachers or Activities

Do you wish to attend a conference for mathematics teachers other than one run by the SMC? Do you have a project you would like to pursue as part of your professional development? Would you like to put on a special event at your school for which an outside speaker or additional materials would be needed?

SMC Bursaries for Pupils to Access Learning Opportunities

Do you have one or more additional needs pupils or students for whom funding is needed to access specific learning opportunities (for example in STEM)?

If you wish to be considered for a bursary (in either category) from the SMC, please email our Treasurer, Dr Alan Walker (alan.walker@uws.ac.uk) for further information.
Introduction

Arguing and proving are daily routines in a professional mathematician’s working life. Therefore these activities should also be emphasized in the mathematical teaching process. Research shows though that learning to prove requires a long-term development that passes through certain consecutive stages.

Beckmann (1997) describes five such stages (Stage 0 to Stage 4) based upon a broad literature foundation (e.g. Piaget, Bruner, Aebli, Lompscher, Galperin, van Hiele etc). The second of these stages (Stage 1) is the focus of this article. Students that argue at this stage tend to use figurative or activity-orientated reasoning.

In the German HeuRekAP project students of two classes initially aged 14 were continuously trained in proving and problem solving over a period of one and a half years by the author. In class C(IT) the training was integrated into the ongoing teaching process (Implicit Training, IT). Class D(ET) received special teaching units in proving and problem solving separate from the ongoing teaching process (Explicit Training, ET). For more detailed information about the modalities of these two training types see Brockmann-Behnsen (2013a and 2014). Two other classes served as control groups.

Mathematical problems were given to the students frequently and their written solutions were collected and analyzed. They showed that there was a certain percentage of solutions with reasoning modalities that obviously correspond to the descriptions of Beckmann’s Stage 1.

It should be hypothesized that (A) this percentage decreases during the observational period as the students get more and more familiar with rather elaborate proof modalities. Furthermore, the percentage should (B) decrease at a higher rate in the trained groups compared to the control groups.

Five stages of proving

Beckmann (1997, p. 22) states that learning to prove is a form of cognitive development. Therefore, it appears reasonable to transfer epistemological theories onto the teaching and learning of proving.

Based on Piaget’s stages of cognitive development, Bruner’s levels of perception, Aebli’s basic forms of learning and other approaches concerning this matter, Beckmann filtrates a five-stage development process of proving as an intersecting set out of these theories (ibid., pp. 22–30). She points out that it is important to undergo the initial stages to successfully reach the higher ones (cf. ibid., p. 34).

Stage 0: Pre-Stage

Mathematical statements are gained by mere imitation and unreflective interaction with the environment. Relations between mathematical figures are recognized, but explanations are only based on given examples.

Stage 1: Activity-orientated, figurative Stage

At this stage there is an insight into the need for generality. The proof is dominated by visual conceptions and imagined or even executed actions like cutting out a geometric figure, folding it, etc. In most cases the argumentative path will imply gaps and its presentation is usually very short but still elaborate enough to allow a general insight into the problem. An important criterion for proofs at this stage is the potential to be complemented to a proof of mathematical rigour. The following problem serves as an example:

Problem: A parallelogram is a quadrilateral whose opposite sides are parallel to each other. 

Prove that the area of the parallelogram is given by \( A = b \cdot h \).

An activity-orientated, figurative proof could appear like this (cf. Figure 1):

If I cut away the left-hand part of the parallelogram shaped as a right-angled triangle and stick it onto its right-hand side I can transform it into a rectangle with side lengths \( b \) and \( h \). The area of this rectangle is given by \( b \cdot h \). The area of the parallelogram must be the same as I haven’t wasted any paper and there are no overlaps.
I will return to this example in the context of Stage 3 to illustrate how to complement it into a proof of more mathematical strength.

**Stage 2: Figuratively concluding stage**

The assurance of generality is the focus of this stage. Even though the construction of such proofs still is rather based on contemplation than on theoretical concepts the reasoning is conclusive. Obvious premises and certain underlying theorems will in some cases not be mentioned. Normally these proofs will be written as a colloquial text rather than in a symbolic form.

**Stage 3: Symbolically deriving stage**

At this stage all steps of the proof are mathematically justified and correctly connected to each other by logical terms. All the premises are explicitly mentioned. Geometry is seen by the student as an example of a deductive theory. The presentation of the proof is carried out in a formal and symbolic way.

I would like to pick up my example for a proof at Stage 1. The proof idea stated there can be formalized due to the characteristics of Stage 3 within a two-column proof scheme. This scheme was introduced to the trained classes in the beginning of their training and it was progressively used by those students. Referring to this proof it should be pointed out that there is a cross-reference to the axiomatic system in lines 5 and 9 which should only be expected at Stage 4.

Besides this, it must obviously be established whether this proof is still valid if the parallelogram's height $h$ lies outside the parallelogram:

**Stage 4: Axiomatically deducing stage**

The problem solver at this stage is aware of the axiomatic system underlying the given problem. His/her proof meets the mathematical demand of rigour. Beckmann (1997, p. 30) emphasizes that it cannot be aspired to reach this stage in secondary school.

**Data basis**

The study has been undertaken in a secondary school in Hanover, Germany from August 2011 to January 2013.
Amongst the tasks given to the students during the HeuRekAP-project were the following two that will be examined in this article:

**Rhombus 1**
A rhombus is divided into two triangles by its diagonal.

Demonstrate that these two triangles are congruent.

Write down all your considerations and arguments step by step.

**Rhombus 2**
A rhombus is a quadrilateral with four sides of equal length.

Given: A rhombus with opposite interior angles $\alpha$ and $\beta$.

Prove: $|\alpha| = |\beta|$.

The following solution of student C24 illustrates my methodological procedure:

![Figure 4](image)

In the first sentence she states that the sides and angles of both triangular parts of the rhombus have the same value – without justifying it. The second sentence could be translated as: They [both triangular parts of the rhombus] could be laid over each other without overlap. Afterwards she examines the given situation from another activity-orientated perspective: one could place a mirror in front of one of the two [triangular parts of the figure] – one would recognize that it looks exactly like the other. For this solution I have coded the verbs to lay (the shapes) over each other and to reflect.

I then determined the percentage of solutions with verbal indication of Beckmann’s Stage 1 for the four classes and the four surveys.

**Results**

Hypothesis (A) could be confirmed by the determined percentages of Stage 1 proofs:

In all four classes these percentages declined in the course of time. The strongest decline could be located in the trained class D(ET): here in PRE2 one third of the students used prove modalities at Stage 1. Only three months later in ICT1 the percentage dropped to 3%, stayed there in ICT2 and went down to 0% in ICT4. The allocated control class A(V.) showed the highest percentage in the beginning: Nearly half of the class used Stage 1 proofs. This percentage fell more or less steadily down to a quarter towards the end of the study. The second trained class C(IT) started with approximately one quarter of the students using proofs at Stage 1 in PRE2. Here the percentage dropped down to 16% within only three months in ICT1. Afterwards it slowly decreased down to 10% in ICT4. An interesting development can be detected in control class B(V.). In PRE2 and ICT1 the percentage remains on the same level (27% and 28% respectively) only to drop down to 6% in ICT2. In the last test not a single student delivered a proof at Stage 1. Retrospectively, I tried to find out...
what happened to class B(V,1) in between ICT1 and ICT2 by talking to the colleague who taught the class and by analyzing the class register. Unfortunately, we were not able to find a substantive reason for this observed development.

Hypothesis (B) could therefore not generally be confirmed because the percentage of students of class C(IT) arguing at a Stage 1 level dropped by 16 points during the observation period and that of D(ET) by 33 points, whereas the control class A(V,1) declined by 23 percentage points and B(V,1) by 27. This demonstrates no tendency in favour of the trained classes.

A noticeable detail of the graphs of the trained classes in Figure 5 though are the prominent edges that can be observed in between PRE2 and ICT1. Both graphs drop down in this period of time and decrease slowly but continuously afterwards.

**Discussion**

Beckmann’s stage model is based on various cognitive development models from the literature. Cognitive development is an individual lifelong process that mainly takes place in everyday life. Research though indicates cultural influences on the timeline of the stages. A mathematical training could be such a cultural influence.

In the Piagetian model, for example, the conversion from Beckmann’s Stage 1 towards Stage 2 corresponds to the conversion from Concrete Operational Stage towards the Formal Operational Stage which is supposed to take place around the age of 11 and marks the conversion from inductive to deductive thinking.

The training units of the HeuRekAP-study conducted in between August (PRE2) and November 2011 (ICT1) comprised of five topics:

1. Auxiliary lines
2. Two-Column-Scheme
3. Notations
5. The Two Gates (cf. Brockmann-Behnse, 2013b)

The dropping of the percentage curves of the trained classes as described in the results section might be

**Figure 5** Percentages of proofs on Stage 1 throughout the study

**Figure 6** The proof of student D02 in ICT4
the consequence of the cultural influence of one or more of these topics. Further research must analyze the causalities. Most likely the use of a Two-Column-Scheme and the obedience of the rule of the Two Gates are capable of triggering the transformation from inductive to deductive thinking. It is remarkable that no proof structured in a Two-Column-Scheme was activity-orientated or figurative. The proof of D03 serves as an example:

"Hilfslinien" means auxiliary lines.
"Stufen-winkelsatz" means "Corresponding Angles Theorem" etc.

In addition to this, some observations I made while analyzing the material seem worth mentioning:

**A) There is no "Cutting Type"**

Students that deliver proofs at Beckmann’s Stage 1 in more than one survey do not usually return to the same mentioned activity. An extreme example is student A33. She delivers Stage 1 proofs in three surveys and described different activities in each of them: in Pretest 2 she intends to turn and to fold the figure, in ICT1 she describes how to reflect one partial triangle of the parallelogram onto the other and in ICT4 she intends to divide the figure.

Of all the students who delivered Stage 1 proofs in two of the surveys 11 described different activities and only four the same. Of all the students who delivered Stage 1 proofs in three of the surveys no one held on to the same activity in all three; six repeated the described activity in two surveys and two (like A33) described different activities in all three surveys.

**B) At the dawn towards higher stages**

Many written proofs are already at Stage 2 or even 3. This should be expected: due to Beckmann’s termination of the attainment of the stages (Beckmann, 1997, p. 22) the majority of the students of the HeuRekAP-training should have reached these stages (cf. Figure 7).

The proof of student C27 is a good example for the definition of Stage 2 given earlier.

Visual aspects still play a central role in the proof. In the first sentence she writes, *If one draws an auxiliary line then one can see that the two triangles (ABC and ACD) are congruent.* More appropriate theoretical reasons for the congruence like the application of congruence postulates for triangles had been an indicator for Stage 3. The next sentence illustrates the conclusive character of this proof: *If two triangles are congruent, the values of the [inner] angles must match.* This sentence also shows that the student had the idea of generality: what she expresses is true for any two given congruent triangles. There are no connections to the premises and the proof is written in colloquial terms rather than in a symbolic way. Her last sentence says: To validate this one could place a mirror onto line CA. Here she returns completely to Stage 1. This phenomenon will be discussed more explicitly in the following section C.

**C) The activity-orientation lives on**

Even students that already argue at higher stages still stick to activity-orientated perceptions. A30 in ICT2 and A31 in ICT4 both follow the same solution path (Figure 9).

They justify the equality of the absolute values of the angles $\beta_1$ and $\beta_2$ with the vertically opposite angle theorem and the equality of the absolute values of the angles $\beta_1$ and $\beta_2$ as well as $\beta_2$ and $\alpha$ with the corresponding angles theorem.

Even though it is completely sufficient to argue that way in mathematical terms, they unnecessarily add activity-orientated reasons to “strengthen” their
argument. A31 states that β can also be reflected by a mirror into β₂. A30 supports his angle-theorem-based argument with imagining angle β, shifting into angle β₂’s position.

Notes

1 The rule of the Two Gates prompts the students to recall two central questions for every mathematical step they undertake: (A) Why are you allowed to do this? (B) What is the benefit of it?

References


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Mathematicians on Coins (2)

A selection of mathematicians who have shaped the subject’s development: Fibonacci, Pacioli, Kepler, Descartes, Newton, Euler, Monge, Lobachevsky, Abel, Abbe, Kovalevskaya, Lyapunov, Voronoi, Russell and Banach. Chris Pritchard
The purpose of this article is to describe the mathematics that emanates from the construction of an origami box. We first construct a simple origami box from two rectangular sheets and then discuss some of the mathematical questions that arise in the context of geometry and algebra.

Mathematical Art and Artistic Mathematics

Many curricular documents in mathematics education recommend that students enjoy exploring and applying mathematical concepts to understand and solve problems, explain their thinking and present their solutions to others in a variety of ways. At all stages, collaborative learning should be emphasised so that students may reason logically and creatively through discussion of mathematical ideas and concepts [1]. Origami provides an engaging context for exploring and applying mathematical concepts and ideas. Traditionally speaking, origami is the Japanese art of folding square sheets of paper into three-dimensional objects; scissors and glue are not allowed.

Among other things, origami gives students handy manipulatives that can be used to visualise abstract mathematical ideas in a concrete way [2, 3]. For instance, when one creates a box from a flat sheet of paper, the box becomes the object that can be manipulated and analysed, and abstract concepts like length, width, height, volume, and surface area become more tangible. When they have objects that they have created in front of them, students communicate better with one another and with their teacher. Moreover, certain aspects of paper-folding can be viewed as mathematics in action. Many folds in traditional origami are inextricably linked to mathematical concepts such as perpendicular bisection, angle bisection, properties of right isosceles triangles and properties of squares. Due to the strong link between origami and art, origami can additionally be used to inspire artistic-minded students to think mathematically. Lastly, origami creates a powerful context for the application of Howard Gardner’s theory of multiple intelligences [4, 5, 6]. Gardner’s theory incorporates several other dimensions of intelligences besides linguistic and logical-mathematical intelligence. He identified the following nine intelligences: linguistic, logical-mathematical, bodily-kinesthetic, spatial, musical, interpersonal, intrapersonal, naturalist and existential intelligence [4].

Origami has come a long way. It is not just about toys that kids make with paper. It has deep-rooted applications in science, technology, and medicine [7, 8, 9]. In this article we will learn to fold an origami box and discuss the mathematics that emanates from the box. The author has asked many of his students to make this box by watching the video (at the links provided below). Some of these students were preservice teachers enrolled in a geometry course designed for teachers, and others were majoring in STEM disciplines enrolled in Precalculus and Calculus courses in a four-year college in the US. Almost all students were successful in making the box without much help from the author.

The box we will fold was designed by the author and requires rectangular sheets of paper; so, this is not traditional origami. No experience in origami is needed to make this box. However, it is important to make the creases sharp and accurate. We will need two identical rectangular sheets of paper. One of the sheets will be used as a measuring tool; we will call this the measurement sheet. The measurement sheet will be discarded. The other sheet will be folded to form the box; we will call this the origami sheet. In order to understand the rest of the article, the readers need to go to one of the following links to watch a video that shows how the box can be folded:

- Moonlight Sonata
  https://youtu.be/a8fHjREsj00
- Jingle Bell Jazz
  https://youtu.be/LjvlGFV33E

The videos at the above links are identical, though their soundtracks are different. Figure 1 shows a picture of the box we will be discussing. In the video we used 210 mm by 297 mm (A4) rectangular sheets. Our final goal is to determine the volume of the constructed box, if we know the dimensions of each of the two identical rectangular sheets. Let us now think about the mathematics that is involved with the folding of this box. Let us carefully unfold the box and look at the crease marks. What is the volume of the constructed box? No, a ruler cannot be used and the volume has to be exact. Readers are encouraged to determine the volume of the constructed origami box before proceeding any further. Let a and b denote the lengths of the shorter and the longer edges of the rectangular sheets, respectively. That is, \( a < b \). For an A4 sheet of paper, \( a = 210 \text{ mm} \) and \( b = 297 \text{ mm} \).
In order to be able to follow the mathematics, the readers must have a sound understanding of how the box was constructed in the video (at the links provided above). Figures 2–5 refer to various stages of the construction process. Figure 2 shows the first four creases that were created on the origami sheet. The rectangle in the middle, bounded by these four creases, forms the rectangular base of the constructed box. To understand where the measurements in Figure 2 come from, we need to dig a bit deeper. Figures 3 and 4 help us do just that.

Figure 3 illustrates how the creases that are parallel to the longer edges of the origami sheet are made. It also helps us explain why the two creases that are parallel to the longer edges of the origami sheet are \( \frac{b}{4} \) mm away from the nearest longer edge. However, Figure 3 only shows how one of the creases that are parallel to the longer edges of the origami sheet was made. The process is identical for the next crease.

Suppose the dimensions of the rectangular base of the constructed box are \( x \) mm and \( y \) mm \((x < y)\), and the height of the constructed box is \( h \) mm (see Figure 5).

As can be gleaned from Figures 2, 3 and 5, due to the nature of the fold,

\[
h = \frac{1}{4} \left( a - \frac{b}{2} \right) \text{ mm.}
\]

From our discussion above, \( x \) and \( y \) must be \( b/4 \) mm and \( b/2 \) mm, respectively (see Figures 2 and 5). Therefore, the formula (in \( \text{mm}^3 \)) for the exact volume of the constructed box must be:

\[
hxy = \frac{1}{4} \left( a - \frac{b}{4} \right) \frac{b}{2} \frac{b}{2} = \frac{b^2}{128} \left( 4a - b \right).
\]
The students who were involved in this activity seemed to enjoy the process of folding a rectangular sheet into a box. Many of the students found the mathematics involved with determining the volume of the box fairly challenging. Even though the mathematics involved is relatively simple, students were new to finding the distance between parallel edges and creases without using a ruler. The mathematical discussions (mostly about geometry and algebra) that ensued were rich and engaging. During this origami lesson there was not a single dull moment in any of the classes; students were actively engaged with their minds and bodies.

References

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SMC Achievement Award 2021

Do you know an exceptional maths teacher or an inspirational leader in mathematics education, someone who has made a significant contribution to mathematics education locally or across Scotland over a period of time?

Then please nominate that person for the 2021 SMC Achievement Award by contacting the SMC Chair, Carol Lyon, by email at carol_smc@icloud.com.
The SMC is currently engaged in the writing of material which will support pupils’ multiplicative reasoning whilst also promoting principles of equity and democracy. Here we offer a glimpse into a small part of the project.

**Fair allocation of seats via simple proportions**

In a democracy, we might expect that a vote cast by one person has the same weight, the same effect, as anyone else’s vote. At the very least we should expect that our elections would include such a design principle. In practice, such fair representation proves to be illusory and impracticable, the mathematician’s best efforts falling short in the face of competing philosophies and systems, and in some cases, in the struggle between political parties to gain an edge, occasionally even by shady manoeuvres.

Imagine that we have five political parties. We won’t give them traditional names because to do so might give the impression of some bias towards a particular party when we simply want to highlight the mathematics. Let’s represent them by five mathematical shapes, Circle, Square, Rectangle, Triangle and Hexagon.

**Example 1**

In this first example of (perfectly) fair representation, let’s keep the numbers small, for convenience. In a tiny ward (voting area) a Council of 20 people is being elected by 100 voters. The number of votes cast for each of the parties is shown in the table below. How many Council members should each of the parties get?

<table>
<thead>
<tr>
<th>Party</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Votes</td>
<td>40 25 5 10 20</td>
</tr>
</tbody>
</table>

To get from 100 to 20, we need to scale down by a factor of 5, which gives 8, 5, 1, 2, 4. These numbers sum to 20, so the calculations are correct. So we have:

<table>
<thead>
<tr>
<th>Party</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Votes</td>
<td>40 25 5 10 20</td>
</tr>
<tr>
<td>Members</td>
<td>8 5 1 2 4</td>
</tr>
</tbody>
</table>

Now 400/25 = 16, so it takes 16 votes to elect a member. Following Example 1, we need to scale by a factor of 1/16 which gives:

<table>
<thead>
<tr>
<th>Party</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Votes</td>
<td>32 88 40 172 68</td>
</tr>
<tr>
<td>Members</td>
<td>2 5½ 2½ 10¼ 4¼</td>
</tr>
</tbody>
</table>

The Circle Party can be fairly represented (by having two members) but this is not possible for the other four parties because the number of members for each of them must be a whole number. In this particular case, the Squares and the Rectangles have many interests (or even properties) in common and they agree to work together. This is also true of the Triangles and the Hexagons. Furthermore, the election rules allow them to decide what to do about the fractions. The Squares and Rectangles agree to have 5 members and 3 members respectively, but the Triangles demand 11 members and the Hexagons eventually agree. The final arrangement sees the Squares and the Hexagons slightly under-represented but it’s not a bad compromise.

**Example 3**

For a third ward, there is a convention that parties cannot enter into the sort of arrangements seen in Example 2. Instead, simple rounding to the nearest whole number is used. There are 377 votes cast to determine the makeup of a 23-member Council. So we need to scale the number of votes cast for each party by 23/377 and then round accordingly.

<table>
<thead>
<tr>
<th>Party</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Votes</td>
<td>42 31 67 100 137</td>
</tr>
<tr>
<td>Members</td>
<td>2.6 1.9 4.1 6.1 8.4</td>
</tr>
</tbody>
</table>

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>
Constituencies, and First Past The Post

In the previous examples the main issue in allocating seats fairly is to distribute fractional remainders. A perfectly fair remedy would be to chop individual politicians into fractional pieces. However, while this may be a popular solution it is not entirely legal. Now we look at another aspect of the division of votes in a representative democracy. In most legislatures, the allocation of seats to parties is done at a fine-grained level. That is, the electoral area is partitioned into smaller regions. These pieces are given many names such as constituencies, wards or precincts. Votes are counted at the constituency level and a certain number of members are sent to the legislature based on the results. Some balance is attempted when partitioning voters so that each constituency represents a coherent community, and that the population of each partition is almost uniform. There are many reasons for dividing an electoral area in this way; in particular it means that voters can contact a representative who is aware of the local issues affecting the constituency.

The rules for creating constituencies differ from country to country and also depend on the type of legislature within a country. In our examples we will simplify matters by assuming that each constituency has an identical population. The number of members each constituency elects also varies in practice. For example, for the elections to the Westminster parliament each constituency provides a single MP while in the Glasgow City Council elections, each ward is represented by either three or four councillors.

Example 4

A council is elected over 5 wards each electing 4 members. The votes cast are as follows.

<table>
<thead>
<tr>
<th>Party</th>
<th>□</th>
<th>□</th>
<th>□</th>
<th>△</th>
<th>□</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ward 1</td>
<td>50</td>
<td>0</td>
<td>25</td>
<td>0</td>
<td>25</td>
<td>100</td>
</tr>
<tr>
<td>Ward 2</td>
<td>20</td>
<td>0</td>
<td>40</td>
<td>5</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>Ward 3</td>
<td>5</td>
<td>0</td>
<td>50</td>
<td>0</td>
<td>20</td>
<td>100</td>
</tr>
<tr>
<td>Ward 4</td>
<td>50</td>
<td>0</td>
<td>0</td>
<td>20</td>
<td>5</td>
<td>100</td>
</tr>
<tr>
<td>Ward 5</td>
<td>25</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>40</td>
<td>100</td>
</tr>
<tr>
<td>Totals</td>
<td>150</td>
<td>0</td>
<td>125</td>
<td>50</td>
<td>50</td>
<td>500</td>
</tr>
</tbody>
</table>

A reasonable allocation of members is given by dividing the votes in each ward by 25 and rounding. Notice that in this case the total number of seats allocated to each party is perfectly proportional to the number of votes they received, namely 25 votes per seat, and these totals match exactly a fair allocation of 20 seats to the 500 total votes.

Example 5

Consider the situation when the results in Example 4 are repeated but that each ward allocates a single member. This member will be from the party which receives the most votes in a ward. The allocation is then as follows.

<table>
<thead>
<tr>
<th>Party</th>
<th>□</th>
<th>□</th>
<th>□</th>
<th>△</th>
<th>□</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ward 1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Ward 2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Ward 3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Ward 4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Ward 5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Is this fair? We might imagine that the Squares and their supporters would be very disappointed. Despite receiving the second most votes overall they won nothing. On the other hand, the strong local support for the Hexagons in Ward 5 means that they will be represented in the legislature. Furthermore, the Hexagon’s councillor will have a casting vote in resolving disputes between the Circles and the Rectangles which in effect makes their single representative the most powerful of all.

There are strong arguments in support of the system characterised in Example 5, known as First Past the Post. For example, it ensures that parties with strong local support are represented (like the Hexagons) and it can reduce the chances of extremist politicians getting enough support to gain representation. Nevertheless, it can lead to results which, at least mathematically, appear unfair and there are frequent calls to replace the system where it still exists. In general, the more seats are allocated per constituency the fairer the final allocation in terms of overall votes, but there are plenty of exceptions to this statement.

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Why learning mathematics is hard ... and what we can do to help

Stuart Welsh

Not all learning is equal

How would you define knowledge? What is learning? Before going any further, I would like to start by defining knowledge as everything that we know (or can do) which is stored in our long-term memory. This allows us to define learning as a permanent change in the content of long-term memory; an accumulation, or a re-organising, of stored knowledge. It should follow then, that learning should be as simple as encountering and remembering knowledge about the to-be-learned topic. Unfortunately, it is not that simple! Some knowledge is acquired more easily than other knowledge and hence some learning comes easily while some learning is hard. For example, consider learning to speak versus learning to write. Our ability to speak has been evolving since the days of the first humans. We are so programmed to learn to speak that even children of profoundly deaf couples turn out to be fluent speakers (Harris, 1998). Writing, on the other hand, is a relatively new cultural invention that has only existed for the past few millennia, which is ‘far too short a time to be influenced by biological evolution’ (Sweller, Ayres & Kalyuga, 2011). Humans are yet to develop the biological mechanisms required to learn to write without instruction.

Primary or secondary

Knowledge can be classified as either biologically primary or biologically secondary (Geary, 1995). Biologically primary knowledge is knowledge which we have evolved to acquire over many generations – often for reasons of survival. It does not usually need to be taught and is acquired with relative ease and speed. Examples include learning to speak, learning to read faces, learning basic social interactions and general cognitive skills such as basic problem-solving or self-regulation (Tricot & Sweller, 2014). In contrast, biologically secondary knowledge is knowledge we have developed for cultural reasons and tends to be acquired relatively slowly and with conscious effort. It has emerged relatively recently in human existence and – as we have not evolved to acquire this type of knowledge – explicit instruction is required for us to learn it. Examples of biologically secondary knowledge include learning to read and write. In fact, almost everything that is taught in schools can be considered biologically secondary knowledge. Therefore, learning in school is hard because it requires effort on the part of the learner and is dependent on the quality of instruction and the quality of learning experiences available.

Learning mathematics is hard

Mathematics may be the most interdependent and hierarchical body of knowledge we expect students to learn. Early experience with counting and numerosity occurs pre-school with most students going on to study mathematics in some form until they are at least 16 years old. That is a lot of content! If we accept that students develop new ideas and understanding by reference to ideas they already know, then the importance of building robust and complex knowledge structures (schema) in long-term memory is critical to successful learning. Poorly-formed schema, lack of relational understanding, misconceptions, or simply missing knowledge, all contribute to lack of progress when learning mathematics.

We must not forget that practice is essential for successful learning in mathematics. However, time spent practising must result in maximum gains. Practice should take advantage of research-based approaches like retrieval, spacing and interleaving. It should also be purposeful and effortful on the part of the learner (Welsh, 2018).

Learning mathematics is hard because it involves concept formation. This is different from learning facts and requires students to develop – for themselves – an understanding of how or why a concept must be so. This cannot be “told”, rather it requires students to use their own awareness to come to “see” the concept for themselves. Learning mathematics is hard because it involves a progressive change in what a student knows (or can do) and relies on a robust understanding of subordinate concepts. Figure 1 is my attempt at exploring the concept of “Adding Fractions”. You can see that the subordinate concepts that connect directly to “Adding Fractions” also have their own subordinate concepts. The concepts around the outside of the diagram will also have their own subordinate concepts, highlighting the deeply structured and interrelated nature of even a relatively simple subordinate concept like adding fractions. I feel this is a worthwhile exercise to carry out when planning a series of lessons as it highlights the key prior knowledge that should be present before the new learning is encountered.

I have met many students who believe mathematics to be a series of unrelated procedures that are to be memorised. Sometimes, that is because
they have been taught that way. Perhaps, this is due to increasingly predictable and formulaic assessments leading to the assessment tail wagging the pedagogical dog. I wonder whether some of our most successful students are, in fact, just very good at remembering processes and recognising when to apply them. Might it be possible that this apparent fluency masks a lack of conceptual or relational understanding? These students may well achieve a top grade come exam time, but are enough really mastering the subject?

Learning mathematics is hard because so much of it exists in the abstract. We can touch, hold and count seven objects, but what about $x$ objects? We can see and feel what it means to add three but it is not so easy to “see” what happens when we subtract negative three. Learning mathematics involves extracting information and making sense of experience. We cannot control what students think but we can try to influence what they think about. Mathematical thought requires “awareness” such that we might form concepts for ourselves before using them in new situations (Wheeler, 2001). I believe that students’ awareness is a powerful and often under-used resource. Indeed, Guy Claxton claims that “the learning mechanism is fuelled by awareness” (Claxton, 1984). The notions of awareness and of teachers ‘forcing’ awareness stem from the fascinating work done by Caleb Gattegno in the 1970s and 1980s. Gattegno tells us that, for students, learning or building knowledge does not happen as the teacher narrates information but rather as students make a conscious effort to focus their attention to develop or “educate” their own awareness (Gattegno, 1987). A learner can educate their awareness by observing what happens in a situation. Here we have an example of two tasks relating to place value.

<table>
<thead>
<tr>
<th>Write down the value of the underlined digit in each number.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) 382</td>
</tr>
<tr>
<td>(c) 93742</td>
</tr>
<tr>
<td>(e) 636324</td>
</tr>
</tbody>
</table>

Figure 2

Figure 2 shows my representation of a standard textbook exercise on place value. The problem with this type of practice is that the learner either gets it correct, i.e. they already know how to do it and probably don’t need the practice, or the learner gets it wrong. If they get it wrong, there is no opportunity for them to educate their awareness.

Contrast this with Figure 3 which shows an adapted version of a task found in the wonderful Practising Mathematics – Developing the Mathematician as well as the Mathematics available from the Association of Teachers of Mathematics (ATM). This task offers some practice of working with place value but with the added benefit of allowing students to observe and make sense of experience. A wrong answer here is useful as students have the opportunity to “see” the effect of their attempt and to take corrective action. In this way, this task not only offers practice but is also a learning opportunity.
Imagine a student who has never heard of a “factor”. The learning experience might proceed as follows:

Teacher: “Today we are learning about factors.”
Teacher: “A factor is... insert teacher’s definition of a factor...”
Teacher: “Here are some examples of factors.”
Teacher: “Now you find all the factors for these numbers.”

By “telling” students a definition of a factor, I think we have missed an opportunity to educate their awareness. Yes, the word “factor” needs to be told (arbitrary), but the meaning or concept can be seen by students. Consider Figure 4. Having said nothing about factors, I would reveal this diagram line by line.

Teacher: “What do you notice?”
Teacher: “Why are some numbers circled?”

As more lines are revealed it is entirely possible for students to come to realise that the numbers circled are all the whole numbers that divide exactly into the “end” number. The students can provide the “meaning” of factor: Leaving the teacher to say: “Yes, and we call these numbers ‘factors’.”

Received wisdom

Teaching is more than the telling of facts and the demonstration of processes. As Elliot Eisner writes, “The aim of the education process... is not to cover the curriculum, but to uncover it”. While it is true that some “telling” must take place to pass on arbitrary knowledge, we must beware of teaching necessary results as if they were arbitrary. By that I mean we should avoid “telling” knowledge and demonstrating results which students can (with our guidance) work out for themselves. If we teach necessary as arbitrary, this knowledge becomes “received wisdom”. Of course, some students will have the awareness to see the “why” behind the result, i.e. they will use their awareness to convert this received wisdom into a necessary result. However, not all students will have the awareness (or even the “bothered-ness”) to do so. Too often, students don’t make sense of the “why” and if necessary results are taught as arbitrary then the received wisdom becomes another unrelated fact to be remembered, or forgotten.
In Figure 6, we see the same approach being used to explore square numbers.

Teacher: "What is special about these rows?"

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
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<tr>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
</tbody>
</table>

*Figure 6*

Developing this use of awareness even further, Figure 7 shows an interesting subset of the original set of rows.

Teacher: "Why have these rows been kept?"

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>3</td>
<td>4</td>
<td>5</td>
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<td>3</td>
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<td>7</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>

*Figure 7*

As you will notice, Figure 7 focuses awareness on the numbers with exactly four factors. This can lead to an investigation into the number of factors between 1 and 100 with exactly four factors.

**Nix the tricks**

It can be tempting to show students shortcuts or tricks – especially the lower-attaining ones. Tricks such as: “KFC” (Keep, Flip, Change) used when dividing by a fraction; FOIL (First, Outer, Inner, Last) for expanding brackets; adding zeros when multiplying by 10, cross multiplication, etc. Consider what has been said about arbitrary and necessary. Tricks are arbitrary and may be forgotten or confused. Have you ever seen something along the lines of (-7) + (-2) = 9 and been told that “two negatives make a positive”? This is learning without awareness; received wisdom becoming a misconception. Tricks do nothing to educate students’ awareness and only fuel the belief that mathematics is a series of procedures to be memorised. See the website www.nixthetricks.com for a guide to avoiding the shortcuts that cut out mathematical concept development.

**Conclusion**

Many students find learning mathematics hard but there is much we can do to support them. Consider the hierarchical and interrelated nature of mathematics when writing courses. Test for and teach (where required) the subordinate concepts before introducing anything new. Beware of teaching necessary as arbitrary. Spend time with colleagues discussing where the necessary content lies in your courses. We cannot control what students think but we can guide what they think about. Through careful task design, through the use of multiple representations, and by offering a variety of experiences to “uncover” concepts and results for themselves, we should aspire to provide opportunities for students to use their own awareness to come to see what is necessary. Anyone who has ever figured out the “why” of a seemingly mystifying mathematical concept will tell you that educating one’s own awareness is an experience in itself and that it is a satisfying one. I believe our job is to give students the chance to experience this satisfaction for themselves. Learning mathematics is hard, but learning mathematics deeply is a truly rewarding experience.

**References**


**Author**


He is @maths180 on Twitter.
What is the Next Number in the Sequence?

Adam McBride

Introduction

When I give a presentation to a group of school pupils, especially at primary or lower secondary levels, I often start by inviting them to guess the next number in a variety of sequences. We begin with simple things like

1, 3, 5, 7, ...
2, 4, 6, 8, ...
1, 4, 9, 16, ...

(As regards the last of the three, on most occasions pupils will get the next term by looking at the differences between consecutive terms rather than observing that they are the square numbers/perfect squares.)

We then move on to things like

1, 1, 2, 3, 5, 8, ... and 2, 3, 5, 7, 11, ...

It is encouraging that many primary pupils have met Fibonacci numbers and prime numbers. Just as they are getting into their stride I will present them with something that involves some lateral thinking, such as the date of the presentation.

In what follows, I shall look at several sequences of non-negative integers that have attracted my attention recently. Some involve interesting mathematics while others are slightly offbeat.

The sequence 1, 2, 4, 8, 16, ...

When I ask pupils the next number in this sequence, the answer I get is always 32, exactly as expected, because they spot a pattern of doubling. I then ask them if there could be a different answer. This is invariably dismissed as being impossible. That leads to consideration of the following problem.

Place n points round the circumference of a circle and draw the chords that join them in pairs. Suppose that no three of the chords are concurrent inside the circle. Into how many regions is the interior of the circle divided?

Remarks:

1. The problem will be familiar to many readers. In the past, it has been given a context such as cutting up a pizza.

2. The second sentence is designed to avoid things degenerating. If all the interior intersections are no longer distinct, we lose regions. This is analogous to what happens in triangle geometry if you allow the triangle to be isosceles or, even worse, equilateral.

Depending on the size of the audience, I proceed in one of several ways but ideally each pupil will have a pencil, a ruler and a sheet of A4 paper with a pre-drawn circle, to avoid the need to issue compasses. Pupils will then place points on the circle one by one, joining each new point to those that are already there and we keep a tally of the number of regions versus the number of points. When \( n = 1 \), there are no other points and hence there are no chords and only one region, the entire interior of the circle. Things get going properly at \( n = 2 \) and we soon get the following table:

<table>
<thead>
<tr>
<th>Number of points, ( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of regions</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
</tr>
</tbody>
</table>

Up to \( n = 4 \) things are obvious from the diagram but at \( n = 5 \) things get a bit more complicated and pupils are encouraged to number off the regions as I go through them on screen, but not to put the numbers in their diagram. (In the course of all this, it is instructive to check up on pupils’ knowledge of words such as quadrilateral and pentagon.) Now the pupils are asked to add a sixth point and to join it to each of the six points. Now, and only now, they start numbering off the regions in their diagrams. Despite the inevitable mishaps, and after a quick check that no bits have fallen on the floor, there emerges a consensus that the correct number of regions is 31, not 32.

Figure 1
At this stage, I tell pupils that, while pattern-spotting is an important part of a mathematician’s toolkit, it is only one stage of a two-stage process. The second stage is showing that the predicted pattern continues and that is where proof comes in, much later in the curriculum. Indeed, the main reason for discussing this sequence here is to present a proof of the correct formula. This is likely to be unfamiliar to some readers and I find it rather elegant. It could be regarded as proof by deconstruction. Essentially, we start with a complicated diagram, such as that for \( n \) points on the circumference and note what happens at each stage. It is convenient to introduce some notation at this stage.

Let

\[ L = \text{the total number of chords (lines inside the circle)} \]

\[ P = \text{the number of points of intersection of chords in the interior of the circle.} \]

(The original \( n \) points on the circumference are not included in \( P \).)

To spot the key idea, suppose we have been removing chords for a while and have arrived at the situation where only 3 remain, as shown in the left-hand diagram of Figure 2 below.

Suppose we now remove the chord \( AD \). We lose 2 points of intersection and 3 pairs of regions merge, reducing the number of regions by 3. This leads us to the middle diagram. Suppose we now remove the chord \( BE \). We lose 1 point of intersection and 2 regions. This gives us the right-hand diagram.

We remove the final chord \( CF \), losing 0 points of intersection (as there are none left to remove) and 1 region. This shows that every time we remove a chord,

\[ \text{(number of regions lost)} = \text{(number of points lost)} + 1 \]

Thus, on removing all the chords one by one,

\[ \text{(total number of regions lost)} = \text{(total number of points lost)} + L \]

(as we have to add 1 \( L \) times). However, at the end, we have lost all \( P \) of the points of intersection and we have only one region left, namely the entire interior of the circle. It follows that, at the start,

\[ \text{number of regions inside the circle} = P + L + 1 \quad (1) \]

This is the key result and all that remains now is to get expressions for \( P \) and \( L \) in terms of \( n \).

A chord is formed by joining two of the \( n \) points on the circle and the number of ways of doing this is given by the binomial coefficient

\[ \binom{n}{2} = \frac{n(n-1)}{2}. \]

To get an interior point of intersection we choose four points on the circle, label them \( A, B, C \) and \( D \) in (say) clockwise order and draw the chords \( AC \) and \( BD \). The number of ways of doing this is another binomial coefficient

\[ \binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{24}. \]

Hence:

\[ \text{No. of regions inside circle} = \binom{n}{4} + \binom{n}{2} + 1. \quad (2) \]

When expanded out, this expression is a polynomial of degree 4 in \( n \), namely,

\[ \frac{1}{24} \cdot (n^4 - 6n^3 + 23n^2 - 18n + 24). \quad (3) \]

However, to do calculations it is much easier to use (2).

Recall that, in general, a binomial coefficient is given by

\[ \binom{n}{r} = \frac{n!}{r!(n-r)!}. \]

In the first instance this only makes sense if \( n \geq r \). However, the number of ways of choosing \( r \) things from \( n \) without replacement is 0 if \( n < r \). So, if we interpret binomial coefficients as 0 where appropriate, we can see what happens in (2) for the first few values of \( n \):

\[
\begin{align*}
  n &= 1: & 0 + 0 + 1 &= 1 \\
  n &= 2: & 0 + 1 + 1 &= 2 \\
  n &= 3: & 0 + 3 + 1 &= 4
\end{align*}
\]

Figure 2
Note for future reference that the expression (3) is the unique polynomial \( p \) of degree at most 4 such that

\[
p(1) = 1, \ p(2) = 2, \ p(3) = 4, \ p(4) = 8, \ p(5) = 16.
\]

These values agree with \( 2^n \) for \( n = 1, 2, 3, 4, 5 \). It is instructive to see why the two sets of values then diverge. By the binomial theorem

\[
(1 + x)^{n-1} = 1 + \binom{n-1}{1}x + \binom{n-1}{2}x^2 + \binom{n-1}{3}x^3 + \binom{n-1}{4}x^4 + \binom{n-1}{5}x^5 + \ldots
\]

If we put \( x = 1 \), we get an expression for \( 2^n \) as a sum of binomial coefficients, namely,

\[
1 + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3} + \binom{n-1}{4} + \binom{n-1}{5} + \ldots
\]

Using the algebraic formula for the construction of Pascal’s triangle, we can combine the second and third terms, as well as the fourth and fifth terms, to rewrite this as

\[
1 + \binom{n}{2} + \binom{n}{4} + \binom{n-1}{5} + \ldots
\]

For \( n \leq 5 \) only the first three terms at most are non-zero and sum to \( 2^n \) in agreement with what we found above, in view of (2). However, for \( n \leq 6 \) the first three terms no longer add to \( 2^n \) and the difference corresponds to the extra terms that do not appear in (2). For \( n = 6 \) we are missing the single term 1, for \( n = 7 \) we lose 6 + 1, and so on.

It is time for something a bit more relaxing!


The sudden jump from 66 to 2000 is probably a shock and suggests that maybe there is something unusual going on that needs a bit of lateral thinking. This is indeed the case. Spell the numbers out in English to get

two, four, six, thirty, thirty-two, ... fifty-six, ..., two thousand and four, ...

Do you notice anything missing? There is no sign of the letter e. It is banned which leads to these numbers being called the eban numbers. It is easy to see that they must all be even since numbers ending in one, three, five, seven and nine will be banned. Also, ignoring the question of how you would spell out in words numbers with millions of digits, let us agree that there are infinitely many eban numbers.

Now how about this sequence?

3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, ...?

At the Mathematical Association Conference for Secondary Maths Teachers held in Stirling in September 2018, Chris Smith mentioned the eban numbers in his presentation. At the end I asked him about similar considerations when the letter o is banned, which would give rise to oban numbers! The first few are listed above.

It turns out that there are only 454 such numbers, the largest being 999 (nine hundred and ninety-nine). There are no more after that as thousand, million, billion, ..., all contain the letter o. Given his amazing creativity, it was no surprise that Chris went into overdrive and got in touch with Oban High School! With help from Iain McLean in the Maths department there, a competition was organised to involve the entire population of that fine town. A large number of A4-sized laminated sheets were distributed all over Oban, not only in schools but in the supermarket, shops and cafés. Each sheet contained a different set of six oban numbers. By trying to track down as many different sheets as possible, people were asked to work out what all the numbers had in common. The winners were two pupils from Oban High School, one in S2 and one in S6. Chris and I had a day out in Oban to present the prizes and give talks to pupils in S1 and S2.

These two sequences form part of a remarkable resource called The On-line Encyclopedia of Integer Sequences, or OEIS (https://oeis.org) which was launched by Neil Sloane in 1964 and is expanding all the time. If your favourite integer sequence is not there, you can nominate it as an addition to the archive.

Just as there is a story attached to the oban numbers, our next sequence also has quite a long story attached, and again it involves Chris Smith. First we need to do some preliminary work.

More on polynomials

We need two properties connected with polynomials with integer coefficients.

Property 1

Given two points in the plane, there is a unique straight line joining them, which we can write (unless the line is vertical) in the form \( y = ax + b \) (or more usually \( y = mx + c \)). Trivially this is a polynomial of degree 1.
Given three points in the plane we can find a unique function \( y = ax^2 + bx + c \) whose graph contains the three points. In general, this will be a quadratic but if the three points happen to lie on a straight line, the coefficient of \( x^2 \) will be zero. We can combine both cases by saying that there is a unique polynomial of degree at most 2 passing through 3 given points.

Continuing in this way and abusing language slightly, we can say that for every positive integer \( k \) there is a unique polynomial of degree at most \( k \) passing through \( k + 1 \) given points in the plane.

**Property 2**

We shall introduce this by means of a couple of simple examples. The idea is to

- evaluate the polynomial at \( n = 1, 2, 3, 4, \ldots \)
- work out the difference between consecutive values (first differences),
- work out the difference between consecutive first differences (second differences),
- work out the difference between consecutive second differences (third differences),

and so on until something interesting happens. We shall call the polynomial \( p \) and use the notation \( \Delta p, \Delta^2 p, \Delta^3 p \ldots \) for first, second, third, \ldots differences.

**Example 1:** \( p(n) = n^2 + n + 41 \).

This turns up regularly in books on number theory as an example of a polynomial that generates prime numbers for a reasonably large number of consecutive values of \( n \) (in this case for \( n = 0, 1, 2, \ldots, 39 \)).

Here is the difference table

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(n) )</td>
<td>43</td>
<td>47</td>
<td>53</td>
<td>61</td>
<td>71</td>
<td>83</td>
</tr>
<tr>
<td>( \Delta p )</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>( \Delta^2 p )</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We see that the second differences are constant (so that third differences would all be 0).

**Example 2:** \( p(n) = n^3 + n + 5 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(n) )</td>
<td>7</td>
<td>15</td>
<td>35</td>
<td>73</td>
<td>135</td>
<td>227</td>
</tr>
<tr>
<td>( \Delta p )</td>
<td>8</td>
<td>20</td>
<td>38</td>
<td>62</td>
<td>92</td>
<td>128</td>
</tr>
<tr>
<td>( \Delta^2 p )</td>
<td>12</td>
<td>18</td>
<td>24</td>
<td>30</td>
<td>36</td>
<td></td>
</tr>
<tr>
<td>( \Delta^3 p )</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We see that the third differences are constant (so that fourth differences would all be 0).

In general, when the above procedure is applied to a polynomial of degree \( k \), the \( k \)th differences are constant.

**Regions in a circle again**

Now let us put these two properties together in the context of our previous sequence derived from regions in a circle. Suppose we have found the values as before for \( n = 1, 2, 3, 4, \ldots \) of \( p \) of degree 4, given by (2) above, through the 5 points \((1, 1), (2, 2), (3, 4), (4, 8), (5, 16)\).

If we know that this quartic \( p \) gives the correct answer in general, we can use a difference table to work out values of \( p(n) \) for subsequent values of \( n \) without using the expression (3) explicitly.

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(n) )</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>31</td>
<td>57</td>
</tr>
<tr>
<td>( \Delta p )</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>15</td>
<td>26</td>
<td></td>
</tr>
<tr>
<td>( \Delta^2 p )</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Delta^3 p )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Delta^4 p )</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We know that, for a quartic, 4th differences are constant, so we can add more 1s in the last row and then work backwards to get values of \( p(n) \) for \( n = 6, 7, \ldots \) as shown:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(n) )</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>31</td>
<td>57</td>
</tr>
<tr>
<td>( \Delta p )</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>15</td>
<td>26</td>
<td></td>
</tr>
<tr>
<td>( \Delta^2 p )</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Delta^3 p )</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**The sequence 0, 0, 2, 6, 12, 21, 34, 51, \ldots**

When I was shown this sequence by Chris Smith in connection with a competition (as described below) I could discern no obvious pattern. There are three Fibonacci numbers in there but that proves to be a red herring. My only idea was to use the 8 values to construct a polynomial of degree at most 7 through the corresponding 8 points and then use the fact that 7th differences would be constant to work backwards, as above, and get the required 9th value. The difference table from the 8 given values is as shown below:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(n) )</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td>12</td>
<td>21</td>
<td>34</td>
<td>51</td>
</tr>
<tr>
<td>( \Delta p )</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>9</td>
<td>13</td>
<td>17</td>
<td></td>
</tr>
<tr>
<td>( \Delta^2 p )</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Delta^3 p )</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Delta^4 p )</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Delta^5 p )</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Delta^6 p )</td>
<td>-1</td>
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<tr>
<td>( \Delta^7 p )</td>
<td>-2</td>
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<td>( \Delta^8 p )</td>
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</table>
Adding an extra 2 in the bottom line and working backwards produces the difference table below.

<table>
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<tbody>
<tr>
<td>p(n)</td>
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<td>0</td>
<td>2</td>
<td>6</td>
<td>12</td>
<td>21</td>
<td>34</td>
<td>51</td>
<td>72</td>
</tr>
<tr>
<td>Δp</td>
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<td>2</td>
<td>4</td>
<td>6</td>
<td>9</td>
<td>13</td>
<td>17</td>
<td>21</td>
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<tr>
<td>Δ²p</td>
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<td>2</td>
<td>2</td>
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<tr>
<td>Δ³p</td>
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<tr>
<td>Δ⁴p</td>
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<td>Δ⁵p</td>
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So, my answer would be **correct but for completely the wrong reason**. The official explanation of the sequence was that you write out the cubes in order

1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...

and delete the units digit!

There is something strange going on here. If you use only the first 7 values of \( p(n) \), you can fit a polynomial of degree at most 6 through the corresponding 7 points but the above procedure delivers 49, not 51, as the 8th number. Similarly, if we take all 9 values in the table immediately above and fit a polynomial of degree at most 8, this gives a value of 102 for the 10th number whereas it should be 100. So, getting 72 by the above method seems to be a complete fluke, unless any readers know better.

The competition referred to above is called Ritangle and details can be found at

https://integralmaths.org/ritangle/

Initially it was for students doing A Level or the IB but, after representations by Chris Smith, it has been extended to students taking Higher or Advanced Higher. It is a team competition and would be ideal for a group of top students who relish a challenge or two (or 25 or more!).

**And finally**

Here are two more sequences for you to puzzle over. The answers can be found on page 86.

1. 149, 162, 536, 496, 481, ...
2. 1089, 146, 53, 34, 25, 29, 85, 89, 145, 42, 20, 4, 16, 37, 58, ...

**Answer to Sudoku**

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<td>1</td>
<td>7</td>
<td>4</td>
<td>9</td>
</tr>
</tbody>
</table>

Chris Pritchard

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Email: a.c.mcbride@strath.ac.uk
In this article I would like to suggest that we reconsider the place of complex numbers in the school curriculum at Advanced Higher level. In particular, I’d like to express a personal view that complex numbers could be used much more routinely elsewhere within existing topics. This might have two positive effects. Firstly, if students use complex numbers routinely to solve other problems, they will become more confident and more competent in answering the complex number questions! This effect has been discussed as the subordination of one topic within another, e.g. [16]. Secondly, complex numbers make connections between subject areas which might otherwise appear disconnected. It is precisely these kinds of deep connections which give pure mathematics its intriguing beauty and leads to some of its effectiveness in solving problems. I hope that using complex numbers more will allow students to see and appreciate these connections, and their importance to the nature of mathematics.

Complex numbers are a capstone topic in school mathematics, at Advanced Higher level and so they sometimes appear not to be really used in any substantial way. Students lack confidence in using complex numbers and struggle to answer elementary procedural questions. For example, the 2015 SQA Advanced Higher Mathematics paper contained the following question.

13. By writing $z$ in the form $x + iy$:
   (a) solve the equation $z^2 = |z|^2 - 4$;
   (b) find the solutions to the equation $z^2 = i(|z|^2 - 4)$.

The 2015 External Assessment Report described disappointing results for this particular question.

In question 13 only a small minority of candidates made progress beyond expanding $z^2$. The almost universally insurmountable hurdles were a lack of understanding of the modulus of a complex number and a failure to compare either real or imaginary coefficients.

Clearly there is a problem here, but I think there is a deeper problem which goes beyond the face-value difficulties students have with what are relatively simple algebraic manipulations. Students’ lack of confidence and understanding can, perhaps, be addressed by more comprehensive use of complex numbers elsewhere within the existing curriculum.

Before I explain this difficulty, and offer a modest and I hope practical way to address it, I’d like to discuss how we arrived at this position in 2019.

1. Historical development of complex numbers

The mathematical community have taken hundreds of years to appreciate the value of complex numbers. Consider, for example, the equation $x^2 + x + 2 = x + 1$. As can clearly be seen from the graphs shown in Figure 1, the quadratic $y = x^2 + x + 2$ does not intersect the line $y = x + 1$. This illustrates why quadratic equations don’t require complex numbers in general. We would not expect solutions to the above equations, which turn out to be $x = \pm i$, and could not interpret the solution geometrically anyway. In fact, there is a rather interesting geometric interpretation as "sibling curves", [11, 12, 13] but appreciation in these interpretations and illustrations is rather recent.

![Figure 1 Equations representing quadratic and linear curves which do not intersect have no real roots](image)

The first point at which complex numbers become necessary is in solving a cubic equation such as $ax^3 + bx^2 + cx + d = 0$. 

Chris Sangwin
If we let
\[ x = t - \frac{b}{3a}, \]
then the original equation is transformed into the "depressed cubic"
\[ t^3 + pt + q = 0. \]
The depressed cubic lacks the quadratic term. Effectively we have moved the inflexion point onto the y-axis to give odd powers of \( t \), and a constant term. A similar process happens in completing the square with a quadratic. If we further manipulate the cubic into the form \( x^3 + 3px - 2q = 0 \) then
\[ x = \sqrt[3]{q + \sqrt{q^2 + p^3}} + \sqrt[3]{q - \sqrt{q^2 + p^3}} \quad (1) \]

This is Cardano’s formula for the roots of a cubic [3]. The discovery of this formula has a very interesting history, and how to derive the formula is well-told elsewhere, e.g. see [11]. The interesting point for us is that the polynomial has three real distinct roots if and only if \( q^2 + p^3 < 0 \). That is, Cardano’s formula inevitably gives rise to complex numbers, even though these turn out to be complex conjugates which ultimately cancel to give the real answer! As Cardano himself remarks [3, p. 220],

So progresses arithmetic subtlety the end of which, as is said, is as refined as it is useless.

For example if
\[ x^3 - 15x - 4 = 0, \quad p = -5, \quad q = 2. \]
Then using these values gives
\[ x = \sqrt[3]{2 + \sqrt{4^2 + 5^2}} + \sqrt[3]{2 - \sqrt{4^2 + 5^2}} \]
\[ = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}. \]
In fact calculating \( \sqrt[3]{2} \) for \( z = 2 \pm 11i \) gives
\[ x = (2 + i) + (2 - i) = 4. \]

Before the twentieth century, Cardano’s formula was part of the standard elementary pure mathematics curriculum in upper secondary school. It has thoroughly fallen out of favour with curriculum designers internationally, and is no longer included in any equivalent of Advanced Higher Mathematics in any international jurisdiction of which I’m aware, including A-Level Further Mathematics and the International Baccalaureate Higher level. Actually, it is not very often that the solution of a cubic is really needed in mathematics, and when needed we could now use computer algebra. However, there is a trap here for the unwary.

Most publications containing the Cardano solution of a cubic equation do not mention that his formula is not always correct for non-real coefficients. Consequently this formula has been misused by many people, including some computer algebra implementers, such as me. The consequences can be disastrous [24].

When calculating the cube roots as part of Cardano’s formula we would expect there to be three complex solutions. Namely, if \( a = \sqrt[3]{2} \) is one root, the three cube roots are \( a, \omega a, \omega^2 a \) where
\[ \omega = e^{2\pi i/3} = \frac{1}{2}(-1 + \sqrt{3}i). \]
Why take this particular root? Well, it leads to a real solution of the cubic! Why not take the others, and what happens when you do? The full story is rather interesting, and a good explanation of which roots to take and why is discussed in [25, Chapt. XII], for example. Computer algebra system (CAS) designers have discussed these problems at length, e.g. [24]. Note that while Cardano’s formula is important historically, and much of the mathematics involved is very interesting and relevant for Advanced Highers, I am certainly not advocating that it is put on the curriculum.

2. Gradual acceptance of complex numbers

Cardano’s work was undertaken in the mid-1540s, see [23]. What Cardano really found was a multi-step method for solving the cubic, rather than the formula written as (1). Indeed, Cardano could not have written \( x^3 + 3px - 2q = 0 \) because algebraic notation, including exponential notation, was not developed for nearly another hundred years. Descartes used the exponential notation we would recognise, and Newton invented rational powers (instead of writing \( \sqrt{x} \), which is older notation) in his work on the binomial theorem, [2]. During the seventeenth and early eighteenth centuries algebra, together with its notation, gradually crystallized into the form we now recognize, see [15, 22]. One purpose of elementary algebra is to classify, and solve, various kinds of equations. The abstraction of problems into an algebraic equation, and then solving that equation, is what [21] refers to as the “Cartesian pattern of thought”. This is one of the most important tools mathematicians have, and this mode of reasoning predominates normative answers to school examination questions. Early elementary algebra textbooks organised these methods. The most important algebra textbook ever written is Euler’s Elements of Algebra, [8]. It is the first algebra book in modern, recognizable form, and is still a thoroughly worthwhile read (although the exercises are strange, see [14]), Euler is unique, in my experience, in trying to introduce and use complex numbers before any real algebra has been introduced. For example, he adds, subtracts, multiplies and divides numbers. The multiplication of numbers such as \( 1 + \sqrt{3} \), including expanding powers, provides plenty of work before any algebraic symbolism in the form of unknown variables, are introduced. Therefore, in the section on powers and roots Euler enters into a discussion...
of complex numbers. But, Euler’s discussion of the problem does not correspond to the contemporary view of how to deal with complex numbers!

§148 Moreover, as \( \sqrt{a} \) multiplied by \( \sqrt{b} \) makes \( \sqrt{ab} \) we shall have \( \sqrt{6} \) for the value of \( \sqrt{-2} \) multiplied by \( \sqrt{-3} \); and \( \sqrt{4} \) or \( 2 \), for the value of the product of \( \sqrt{-1} \) and \( \sqrt{-4} \). Thus we see that two imaginary numbers, multiplied together, produce a real, or possible one. [8]

It was De Morgan, who took an alternative view.

The only case which requires notice is that of forming the symbol which is to represent \( \sqrt{-a} \times \sqrt{-b} \). This by common rules, may be either \( \sqrt{-a} \times \sqrt{-b} \) or \( \sqrt{ab} \), or \( \sqrt{a} \times \sqrt{-1} \) multiplied by \( \sqrt{b} \times \sqrt{-1} \) or \( \sqrt{ab} \times -1 \), that is \( -\sqrt{ab} \). For reasons hereafter to appear, let the student always take the later, that is let \( \sqrt{-a} \times \sqrt{-b} \) be \( -\sqrt{ab} \) not \(+\sqrt{ab} \) [6, p. 122].

Did Euler really not understand complex numbers? This is unlikely, especially given his other writing on the subject such as his fascinating discussion and criticism of the arguments surrounding the invention of complex logarithms, see [10], and the original sources [7, 9]. That some of the best mathematicians, including Euler, Leibniz and Bernoulli, were discussing the confusing nature of complex numbers and complex logarithms should give us some comfort. Euler puts the problem rather succinctly.

If at times this disagreement is not expressed strongly the reason is clearly that people do not want the certainties of pure mathematics in general to come under suspicion by revealing in public the difficulties and even contradictions that mathematicians find in this area [10].

It has taken a long time for the mathematical community to understand the rather deep ideas involved with complex numbers, and to develop robust ways to discuss them. This discussion includes the subtle difference between the notation \( \sqrt{-2} \) and \( i\sqrt{2} \). This very small, trivial even, difference actually avoids precisely the kind of computational trap into which Euler falls above. That is, if we immediately write \( \sqrt{-2} \) as \( i\sqrt{2} \), and do not use a minus sign under the square root symbol, then it is much harder to end up with formal manipulations such as

\[
\sqrt{-3} \times \sqrt{-2} = \sqrt{-3} \times -2 = \sqrt{6}.
\]

It is hardly surprising that school textbooks would therefore warn students of the difficulties.

The student is recommended to have as little as possible to do with imaginary quantities, that is, with quantities which have no meaning either as to number or magnitude. He need not wonder that the difficulties are likely to be introduced by the use of them, when he considers that \( \sqrt{-1} \) signified an operation to be performed which is absolutely impossible. [19, p. 75]

The textbook by Lund, [19], had at least 17 editions between 1795 and 1876, a period of 81 years. The original work was written long before the pioneering work on complex analysis in the 1830s by Cauchy, Riemann and others. During the middle and later part of the 19th century school textbooks gradually included complex numbers. It is not really surprising that it took generations for the pure mathematical research of complex numbers to find applications. Some of the most compelling applications occur in the theory of electromagnetism, which started with the work of James Clerk Maxwell in the 1860s. Confident and comprehensive elementary textbook sections on complex numbers, for school students, really only become evident at the start of the twentieth century, with a particularly significant textbook by Chrystal, one of my predecessors in Edinburgh. [4,5]. A hundred years later complex numbers have proved their value, both in pure mathematics and in applications.

3. SQA Advanced Highers

As a high-point at which to end school mathematics Euler’s formula,

\[
e^{i\theta} = \cos \theta + i \sin \theta
\]

together with some immediate applications, is difficult to beat.

![Figure 2](image)

![Figure 2](image)

The utility derives from the observation that every complex equation says three things at once: one complex, one real and one imaginary.

The shortest path between two truths in the real domain passes through the complex domain. Jacques Hadamard (The Mathematical Intelligencer, v. 13, no. 1, Winter 1991.)

The current Advanced Higher specification, relevant excerpts of which are reproduced in Appendix B, just stop slightly short of including Euler’s formula within the assessment topics.

Learners could be exposed to the exponential form of a complex number, \( z = re^{i\theta} \). The Maclaurin series for \( e^z \), \( \cos \theta \) and \( \sin \theta \) can be used to show that \( e^z = \cos \theta + i \sin \theta \) and hence that \( e^{i\theta} = -1 \).
While the use of Euler’s formula (2) is not included in the possible assessment topics, de Moivre’s theorem is included

\[(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta\] (3)

The rest of this article provides some examples and proposals of how complex numbers can be used to prove and derive many other results already part of Advanced Higher Mathematics, without adding any extra material, or significant computational complexity, to the existing curriculum. Indeed, this opportunity is already acknowledged in the course specification: “The proof of de Moivre’s theorem for positive integers should be covered as an example of proof by induction.” For completeness a proof by induction is included in Appendix C.

4. Loci

For school students one of the most important uses of complex numbers is to link algebra, trigonometry and geometry, and this can be done through a vector interpretation of a complex number \(z = a + ib\) as a point in the plane \((a, b)\) and through the position vector \(\begin{pmatrix} a \\ b \end{pmatrix}\). Then algebraic operations, such as addition and multiplication, can be interpreted as a geometric transformation. Multiplication of \(w\) by a real number \(a\), for example, is a stretch by a factor of \(a\). Multiplication by \(i\) gives \(\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} -b \\ a \end{pmatrix}\). This is a rotation 90° anti-clockwise, see Figure 2.

Now consider multiplication of \(w\) by \(z = a + ib\), i.e. \(zw = (a + ib)w = aw + ibw\).

We have added the “vector” \(aw\) to the rotation of vector \(bw\) by 90° anti-clockwise. By drawing a picture of this right-angle triangle, and appealing to geometry, we can prove that multiplication of \(w\) by \(z = a + ib\) is a scaling of \(w\) by \(|z|\), and a rotation by \(\arg(z)\). Hence, if \(z, w \in \mathbb{C}\), then

\[|zw| = |z||w|\] and \(\arg(zw) = \arg(z) + \arg(w)\).

The power of technology to visualize is tremendous. For example, Figure 3 shows the popular GeoGebra software. By looking at the properties of a point, under Algebra, the point can be interpreted as a complex number in the plane.

Most contemporary geometry software will allow complex number demonstrations, but the Cinderella software https://www.cinderella.de/ puts complex numbers at the heart of the internal representation [17]. This immediately allows some simple and compelling demonstrations to be undertaken in class.

1. Define a point \(A\) on the unit circle.
2. Define \(A' = A^2\).
3. Move \(A\) slowly around the circle, and watch as \(A'\) moves at twice the angular speed.
4. Detach \(A\) from the unit circle. Move \(A\) around the origin and observe how \(A'\) moves.
5. If \(A\) moves around a general circle in the complex plane, what is the locus of \(A'\) (in qualitative terms)?

This kind of visual geometrical demonstration suggests how to find the complex square roots of a complex number. That is, given a point \(A'\) in the complex plane, for which points is \(A^2 = A'\)? There are obvious generalities such as points \(A^n\), together with the \(n\)th roots. When visualized by moving points around, the impossibility of creating a continuous square root function on the complex plane leads onto the difficult subject of branch cuts. While this is not something Advanced Higher students might dwell on, planting the seed that a real and
inescapable difficulty exists here is valuable for the future. A second example of Loci, taken from [20], is to consider the differential equation

\[ \frac{dz}{dt} = iz, \]

where \( z(t) \in \mathbb{C}. \) Yes, here we have \( i. \) Proceeding (with some confidence) with pure calculus we notice this is a separable differential equation (within the AH curriculum) and separating variables we get

\[ \int \frac{1}{z} dz = \int i dt \]

from which

\[ z(t) = z_0 e^{it}. \]

This is only part of the story, since when interpreted geometrically multiplying by \( i \) is a 90° rotation anticlockwise. We can therefore read the equation

\[ \frac{dz}{dt} = iz \]

as “the velocity vector is at right angles to the current position vector; and the magnitude of the velocity equals the distance from the origin”. This leads to circular motion! Therefore, the solution we have, \( z(t) = z_0 e^{it}, \) moves around a circle in the complex plane. From this discussion we get an informal argument as to why \( e^x = \cos x + i \sin x. \)

5. Proving trig formulae

The following already appears as part of the specification under

Apply de Moivre’s theorem to multiple angle trigonometrical formulae.

e.g. express \( \sin 5\theta \) in terms of \( \sin \theta. \)

e.g. express \( \sin 5\theta \) in terms of \( \sin/\cos \) of multiples of \( \theta. \)

Euler’s formula (2) enables us to prove most of the trigonometrical identities in a very systematic and straightforward way. Expressions with trigonometrical functions have two important forms. The first uses powers of a base angle, e.g. \( \cos^n \theta. \) The second uses multiple angles, e.g. \( \cos n\theta. \) It is often necessary to convert from one form to another, e.g. to enable efficient integration. Squaring both sides of (2) gives

\[ \cos 2\theta + i \sin 2\theta = e^{2i\theta}, \]

\[ = (e^{i\theta})^2 = (\cos \theta + i \sin \theta)^2. \]

If Euler’s formula is not available, then de Moivre’s theorem (3) can be used directly to get

\[ \cos 2\theta + i \sin 2\theta = (\cos \theta + i \sin \theta)^2; \]

In either case, expanding the right-hand side and equating imaginary parts gives

\[ \sin 2\theta = 2 \sin \theta \cos \theta. \quad (4) \]

Similarly we could equate real parts to get a formula for \( \cos 2\theta, \) or use a higher power to derive formulae for other multiple angles. By using (2) with \( e^{i\theta} + e^{-i\theta} \) we can see that

\[ \cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}). \]

Taking higher powers, leads to formulae which express \( \cos^n \theta \) in terms of multiple angles.

6. Integration

I have shown how trigonometrical formulae can be replaced by complex exponentials and in many cases this leads to more direct computation. Just one example, the integral \( \int \cos \theta \sin \theta \, d\theta \) can be calculated (i) by parts, (ii) by substitution, or (iii) by recalling (4) and first making the substitution

\[ \cos \theta \sin \theta = \frac{1}{2}\sin 2\theta. \]

Complex exponentials can also be used directly in place of the trigonometrical formula.

\[ \int \cos \theta \sin \theta \, d\theta = \int \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \, d\theta \]

\[ = \frac{1}{4i} \int (e^{i\theta} - e^{-i\theta}) \, d\theta \]

\[ = -\frac{1}{8}(e^{i\theta} + e^{-i\theta}) + c. \]

Since it is standard practice to rewrite the answer in the form given in the statement of the question, we should use (5) to transform this back into a final answer of

\[ -\frac{1}{4}\cos 2\theta + c. \]

7. Fundamental theorem of Algebra

The fundamental theorem of algebra says that every non-constant single-variable polynomial with complex coefficients has at least one complex root. By induction we expect polynomials of degree \( n \) to have \( n \) (possibly repeated) roots. A consequence of this is that by working over the complex numbers
instead of just the real numbers polynomials factor into \( n \) linear factors, and we have no irreducible quadratic terms. This simplifies a lot of the elementary mathematics encountered at Advanced Higher level by removing special cases for irreducible quadratics. We now return to the differential equation

\[
\frac{dz}{dt} = iz,
\]

which is really one that is very well studied. Differentiating with respect to time gives

\[
\frac{d}{dt} \frac{dz}{dt} = i \frac{dz}{dt},
\]

so that

\[
\frac{d^2z}{dt^2} = iz = -z.
\]

This is the equation for simple harmonic motion. By always using complex numbers here we remove special cases so that

\[
x(t) = Ae^{i \alpha t} + Be^{i \beta t},
\]

works for real and complex \( \lambda \) ! However, the coefficients might now be complex conjugates, and we still have the repeated root case which is rather subtle. Note, the solutions of second order differential equations are included in Advanced Higher Mathematics.

8. Partial fractions

There are many situations where the possibility of having an irreducible quadratic causes awkward special cases in implementing methods. Partial fractions over the complex numbers are given in [19]. We now return to the example

\[
\frac{1}{1+x^2} = \frac{1}{2} \left( \frac{1}{1+ix} + \frac{1}{1-ix} \right).
\]

Assume that \( p(x) \) and \( q(x) \) are co-prime, and that the degree of \( p \) is less than the degree of \( q \). If we are trying to write \( p(x)/q(x) \) in partial fraction form, then being able to write \( q(x) \) as a product of powers of distinct linear factors considerably simplifies the process. Partial fractions over the complex numbers requires fewer cases than over the real numbers. For example

\[
\int \frac{1}{1+x^2} \, dx = \frac{1}{2} \int \left( \frac{1}{1+ix} + \frac{1}{1-ix} \right) \, dx.
\]

This particular partial fraction decomposition can be used to find the integral of \( \frac{1}{1+x^4} \) with respect to \( x \).

\[
\int \frac{1}{1+x^4} \, dx = \frac{1}{2} \left( \ln(1+ix) - \ln(1-ix) \right).
\]

Therefore, admittedly playing fast and loose with the complex logarithm,

\[
\ln \left( \frac{1+ix}{1-ix} \right) = 2iy, \text{ so that } \frac{1+ix}{1-ix} = e^{2iy}.
\]

Solving for \( x \) we have

\[
ix(e^{2iy} + 1) = e^{2iy} - 1
\]

which, using (5), leads to

\[
x = \frac{1}{i} \frac{e^{2iy} - 1}{e^{2iy} + 1} = \frac{1}{i} \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{2}{2i} \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{\sin y}{\cos y} = \tan y.
\]

i.e., \( x = \tan y \), so that \( y = \tan^{-1} x \). Substituting this value of \( y \) in (6) we have

\[
\int \frac{-1}{1+x^2} \, dx = \tan^{-1} x.
\]

Using complex numbers in this way provides an elementary justification of a result which is otherwise simply stated in a table of integrals. This derivation is not suitable for a school examination. However this argument is close to the historical discovery of this integral, see [23, pp. 293–299].

9. Conclusion

The topics in the Advanced Higher Mathematics curriculum are basically sound. Indeed, I would say that no curriculum change is needed. However, I do think we might better employ the curriculum we have to use complex numbers to tie together other topics. In particular, if we accept that students “could be exposed to the exponential form of a complex number, \( z = re^{i\theta} \), and make modest use of (2) then a lot more becomes possible even if some of this won’t appear on examination papers as assessment topics. Complex numbers need not be left as an isolated subject. There are many interesting problems across the levels of difficulty needed for Advanced Higher, including relatively simple work through to more demanding tasks. Further examples and applications of complex numbers are given in [1] and the opening chapters of [20]. By judicious and cautious use of these examples we can raise students’ confidence in using complex numbers.

There is no shortage of potential examination questions, including some rather predictable work which is necessary for a paper which tests students of every ability level. I am not suggesting we make the subject harder, but I am suggesting we make
better use of what we have. Complex numbers is a beautiful and intriguing subject – indeed it is a high point of human intellectual achievement – and we have an opportunity to let our students start to discover this area of mathematics more fully.

**A Practical suggestions for the classroom**

[1] contains a large number of exercises and further examples, many of which will be applicable for Advanced Higher mathematics students. The following suggestions might be useful for projects or extension work. However, they (rightly) go beyond the current curriculum.

- Superimpose the Argand plane onto the Cartesian plane. On the Cartesian plane sketch a typical quadratic (with and without real roots) and on the same plane sketch the position of the roots (as real and complex numbers). If using dynamic geometry, the quadratic can be moved dynamically to explore how the roots also move. See [11] [12] for more suggestions.
- Work through the derivation of Cardano’s formula, e.g. that given in [25, Chapt. XII].
- Consider the search for a definition of complex logarithm, as discussed in the original sources [7,9]. A summary is given in [10].
- Investigate number theory associated with the Gaussian integers \( n + im \). For example, in the set \( \mathbb{Z} \) we have the number 2 is prime. Consider \( (1 + i)(1 - i) = 1^2 - i^2 = 2 \). We consider the complex numbers \( w = 1 + i \) and \( z = 1 - i \) as single entities, i.e. as a number with a representation \( 1 + i \). Therefore, the prime number 2 appears to have two factors: 2 = \( wz \).

**B Advanced Higher Specification**

Section 1.3 from the Advanced Higher Mathematics Course/Unit Support Notes of May 2016, version 2.2 is reproduced in Figure 4 below. Note that assessment standards are indicated by a diamond bullet point. Those skills marked by a diamond bullet point and those marked by an arrow bullet point can be assessed in the course assessment.

**C Proof of de Moivre’s Theorem**

The specification says that the proof of de Moivre’s theorem for positive integers should be covered as an example of proof by induction. This section discusses that proof. A proof by induction has the following formal structure.

1. Let \( P(n) \) be the statement “...” for all \( n \in \mathbb{N} \).
2. Proof of \( P(1) \).
3. Assume \( P(k) \) is true, and prove \( P(k + 1) \).
4. Conclusion: since \( P(1) \) and \( P(k) \Rightarrow P(k + 1) \) for all \( k \in \mathbb{N} \), it follows that \( P(n) \) is true for all \( n \in \mathbb{N} \).

<table>
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<th>1.3 Applying geometric skills to complex numbers</th>
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<tr>
<td>Sub-skill</td>
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<td>Performing geometric operations on complex numbers</td>
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Technology can be used to investigate loci.

![Figure 4](Advanced Higher Mathematics Course/Unit Support Notes)
This basic proof structure remains valid within a number of possible variations. For example, step 2 might prove \( P(k) \) for some \( k > 1 \), in which case the conclusion only holds for \( n \geq k \). Similarly, step 2 might start at \( k = 0 \) (which may or may not be defined as a natural number), or some other integer. In step 3, it is possible to assume \( P(n) \) is true for all \( n < k \) and use any of these statements to prove \( P(k + 1) \), which is a form of "strong induction". However, it is simply not sufficient to show a pattern holds for the first few cases and then assert a general conclusion. See, e.g. [18].

Step 1. For \( n \in \mathbb{N} \), let \( P(n) \) be the statement that

\[
(cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.
\]

Step 2. \( (\cos \theta + i \sin \theta)^2 = \cos \theta + i \sin \theta \), so that \( P(1) \) is true.

Step 3. Assume \( P(k) \) is true. Using the induction hypothesis that \( P(k) \) is true, consider

\[
(cos \theta + i \sin \theta)^{k+1} = (cos \theta + i \sin \theta)^k (cos \theta + i \sin \theta) = (\cos k\theta + i \sin k\theta)(\cos \theta + i \sin \theta).
\]

Next expand the brackets using the standard trig formulae for \( \cos(A + B) \) and \( \sin(A + B) \) and collect real and imaginary parts to give:

\[
\begin{align*}
&\cos k\theta \cos \theta - \sin k\theta \sin \theta \\
&\quad + i (\cos k\theta \sin \theta + \cos \theta \sin k\theta) \\
&\quad = \cos (k+1)\theta + \sin (k+1)\theta.
\end{align*}
\]

In summary this proves that

\[
(cos \theta + i \sin \theta)^{k+1} = \cos (k+1)\theta + i \sin (k+1)\theta
\]

and hence \( P(k+1) \) is true.

Step 4. Now since \( P(1) \) and \( P(k) \Rightarrow P(k + 1) \) for all \( k \in \mathbb{N} \), it follows that \( P(n) \) is true for all \( n \in \mathbb{N} \).

As we have seen, this proof makes use of the standard trigonometric formulae for \( \cos(A + B) \) and \( \sin(A + B) \). One proof of these formulae uses Euler’s formula (2). Indeed, I’m advocating precisely this approach to trigonometric formulae since I think it is the most direct approach, and the same approach can be used for a wide variety of trig formulae. But if we have (2) available to prove trig formulae then we could use (2) for a direct proof of de Moivre’s Theorem, without induction! One solution to this is to independently prove the following, by direct geometric arguments in the complex plane:

1. \( z = \cos \phi + i \sin \phi \) if and only if \( |z| = 1 \) and \( \arg(z) = \phi \).
2. If \( z, w \in \mathbb{C} \) then \( |zw| = |z||w| \) and \( \arg(zw) = \arg(z) + \arg(w) \).

We can then prove de Moivre’s Theorem by induction without the trig formulae. Of course, the above two results would enable us to derive the trig formulae as well.

1. For \( n \in \mathbb{N} \), let \( P(n) \) be the statement that

\[
(cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.
\]

2. \( (\cos \theta + i \sin \theta)^2 = \cos \theta + i \sin \theta = \cos 1\theta + i \sin 1\theta \),

so that \( P(1) \) is true.

3. Assume \( P(k) \). Consider \( P(k + 1) \):

\[
(cos \theta + i \sin \theta)^{k+1} = (\cos \theta + i \sin \theta)^k (\cos \theta + i \sin \theta) = (\cos k\theta + i \sin k\theta)(\cos \theta + i \sin \theta).
\]

Let \( z = \cos k\theta + i \sin k\theta \); then \( |z| = 1 \) and \( \arg(z) = k\theta \).

Let \( w = \cos \theta + i \sin \theta \); then \( |w| = 1, \arg(w) = \theta \).

Then \( |zw| = |z||w| = 1 \), and \( \arg(zw) = \arg(z) + \arg(w) = (k + 1)\theta \).

So \( zw = \cos (k + 1)\theta + i \sin (k + 1)\theta \), giving \( (\cos \theta + i \sin \theta)^{k+1} = \cos (k+1)\theta + i \sin (k+1)\theta \).

Hence \( P(k + 1) \) is true.

4. Since \( P(1) \) and \( P(k) \Rightarrow P(k + 1) \) for all \( k \in \mathbb{N} \) it follows that \( P(n) \) is true for all \( n \in \mathbb{N} \).

There is real potential in all of this for discussion of the chicken and egg potential in pure mathematics, and for students to see the need for a careful path through all this, and the potential for circularity here reinforces the view that de Moivre’s Theorem and Euler’s formula lie at the heart of pure mathematics and enable connections between a wide variety of topics to be made.

References


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Answers to the puzzles in ‘What is the next number in the sequence’ by Adam McBride

1. Write out the perfect squares in a single line with no gaps or commas between them. Then group the digits in threes. The next term will be 100.

2. Those who have heard some of my talks in the past will know that 1089 is one of my favourite numbers, so it is fitting to include it as the starting point of a sequence. In each case here you add the squares of the digits in one term to get the next one. So, the next term after 58 is $5^2 + 8^2 = 89$. You will notice that 89 appears earlier in the sequence, so after 58 the terms go into a loop and we get … 58, 89, 145, 42, 20, 4, 16, 37, 58, 89, …

If we start instead with 31 we get successively 31, 10, 1, 1, …

Readers might like to explore what happens for other starting values.
The reviews below have been written by Jenny Wood, Alan Walker and Karen Hart. Note that the prices given are the full prices quoted by the publisher and that special deals and discounted prices may be available from either the publisher or the retailer.


This book is a colourful summary of the key skills and techniques in the Higher Mathematics syllabus. The design is eye-catching, with examples, key points and "hints and tips" clearly delineated with colour coded boxes and icons.

An introduction covers the key skills from the National 5 Mathematics syllabus necessary for success at Higher level. The book is then organised into four broad sections (algebra, geometry, trigonometry and calculus) followed by a practice paper. For each topic, an explanation is given of the requisite core skills followed by one or two worked examples. Explanations are succinct although this naturally leads to the omission of detail that may be beneficial to learners. For example in Chapter 13 nature tables are shown but no mention is made of substituting appropriate values into the derived function.

Answers are provided for the short exercises at the end of each chapter, and there are worked solutions for the practice paper although an indication of where individual marks are awarded may benefit learners. Similarly, it would be beneficial throughout the book to place more emphasis on rigour and indicate to learners the key items that will accrue marks.

This book is suitable for candidates who are looking for a "checklist" of skills for their exam preparation. It does not offer sufficient practice material to form the basis of a revision programme, but if used in combination with a textbook or practice question book it could well be of assistance to learners.

*Jenny Wood*


This textbook is an engaging and accessible resource for the Higher Mathematics course. Chapters are presented in an intuitive and readable format. Each chapter begins with a list of learning intentions and states the requisite prior knowledge from National 5 before a brief retrieval exercise to ensure learners are ready to proceed. Clear worked examples are then given with essential points highlighted in yellow.

Within an exercise, learners are provided with sufficient practice material to consolidate core skills before tackling more challenging problems. Certain key questions are marked 'ACE' (Application, Communication, Enquiry) or 'HPQ' (Hinge-Point Questions). These are designed to give an indication of whether learners have mastered the foundations of the topic sufficiently before proceeding to the next part of the exercise. A 'checkout' section follows each chapter with more questions to enable the student to monitor their progress. Answers are provided for all questions at the back of the book, including graph work. The textbook is colourful, user friendly and a useful addition to any Maths classroom.

*Jenny Wood*

**Primary Maths for Scotland Textbooks** by Antoinette Irwin, Carol Lyon, Kirsten Mackay, Felicity Martin, Scott Morrow. Leckie and Leckie (2019).

There has been a huge amount of work, carried out locally and nationally, aimed at addressing the nation’s attitude to maths and exploring the way we construct mathematical ideas. For converts of teaching in a conceptual way there has been an overwhelming selection of materials from around the globe. The arrival of textbooks which aim to support deep understanding and are designed for the Scottish curriculum is bound to generate a great deal of interest.

*Second Level Textbooks*, £11.99

There are similarities across levels with each level having three textbooks to support learning. Furthermore, each textbook is structured in the same way, with clear contents pages, an accessible font and answer pages at the back. However, the meaty part, the important part, is of course the content of each section which is further broken down into subsections. Each subsection starts with a learning intention and a thought-provoking 'Before we start' task to assess prior knowledge. These tasks encourage dialogue around the main idea and support teachers in gauging where children are in their thinking. The 'Let's learn' section introduces new concepts and can be used to support teaching whilst the 'Let's practise' section is self-explanatory.
Each subsection concludes with a ‘Challenge’ to extend and stretch as required.

The visual representations that accompany the text are, for the main, clear although some of the finer details on images such as the clocks are less successful. The use of bar models, graphic organisers and number lines offers a variety of representations to explore.

There are some teething problems in terms of answers not matching questions but there can be little doubt that these textbooks, if used judiciously, can be a safe and reliable resource that truly does aim to get children talking and thinking about their maths.

First Level Textbooks, £9.99
1A ISBN 978-0-0083-1395-1
1B ISBN 978-0-0083-1396-8

The First Level textbooks arrived on the scene a little later than the Second Level books. That may be the reason that they have additional features. Colleagues working at First Level are impressed by the free downloadable resources that go along with the books. They are a mix of general resources such as blank ten frames and number lines and specific resources that relate to a specific area of learning.

The books are structured in the same way as Second Level and there are high quality visuals throughout. There is evidence of questions that seek to uncover misunderstanding and many tasks are designed to require a level of understanding to succeed. This may seem obvious but careful examination of some textbooks can show tasks that can be completed successfully without conceptual understanding.

Overall, these textbooks are a welcome addition to any classroom. The teacher guides will be much anticipated to further support teachers in making sure that no child progresses through their schooling with an increasing dislike of mathematics!

Karen Hart


The Room in the Elephant is the second in a series of books written by Dr Chris Pritchard for The Mathematical Association, focussing on the geometry and area of common shapes. It is, according to the back cover, written to be accessible to an able and interested 18-year old, aims to appeal to students and teachers of mathematics and to anyone with a fascination for the subject.

The content includes some investigations into area using tangrams, polygons, tiling, perimeters, and circles. For some, it will be a pleasant walk down memory lane to geometry lessons at school. Others may be found to be questioning why forays into geometry are no longer part of the Scottish mathematics curriculum, at least not to any depth which might arouse curiosity in the subject. For this reader, some of the content reminded him of investigations carried out in primary and secondary, and a dawning of what the teachers were trying to achieve at the time.

As a resource for secondary school teachers, the book provides some excellent ideas for delving further into the straight line, triangles, circles and the relationships between the three. Those who finish material which addresses “applying algebraic skills to rectilinear shapes” or “applying algebraic skills to circles and graphs” before the summer might consider using material contained within this book for deep-diving in the last days of term. This could be particularly useful for student teachers given the opportunity to present something a little off-the-wall, or for those schools lucky enough to have extra-curricular maths sessions. As a resource for university educators, the book provides some great starters for some thorough investigations into geometry.

The book is full of little concepts and terms which this reader had either long forgotten or never knew, such as a Newton Line, Anne’s Theorem, and Heron’s formula. Readers will note many common elements of the shapes and tilings discussed with some recent cover art on both the SMC Journal and the SMC Primary Journal. Further, the realisation—a reader of the SMC Journal and attendee of the SMC conference, really was a facepalm moment.

Pritchard’s style is fun, as can be seen from the wordplay in the title and one or two groan-worthy puns. The book has a little smattering of history lessons throughout, often giving a good introduction to a particular topic, or some colour to a name you may have heard before (see Georg Pick).

In short, this is a thoroughly enjoyable book, with loads of examples and ideas to take into the classroom, lecture, or to any collection of enthusiastic individuals.

Alan Walker