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Fuzzy-based parameter uncertainty in an elastoplastic model for clay

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ABSTRACT: In this paper, we try to analyse uncertainty in constitutive modelling of clays. The main objective is to analyse parameter uncertainty in an elastoplastic model for clay by considering a fuzzy-based distribution of the elastic bulk modulus K and shear modulus G. Interval arithmetic is fundamental in the fuzzy-based approach, as it allows taking into consideration the uncertainty that arises from K and G and thus in the matrix of material properties **D**. A Mohr-Coulomb soil model is considered in the analysis. Fuzzy sets are well suited for representing vague and scarce data encountered in soil tests. Histograms of probabilities can be used to construct fuzzy sets. Simulations are run based on existing data, and comparisons with available results are highlighted. The effect of fuzzy-based parameter uncertainty is quantified using the Hartley-like measure.

Keywords: Parameter uncertainty; Elastic modulus; Mohr-Coulomb model; Elastoplasticity; Fuzzy sets.

1 INTRODUCTION

Elastoplastic theory was adopted in soil modelling earlier in the 50' of the last century (Drucker and Prager, 1952). The theory describes well the mechanical behaviour of geomaterials in general. Elastoplasticity-based models consider soils as a continuum for deriving usable constitutive relations which seem to accommodate their actual mechanical behaviour. However, soils are inherently discontinuous and carry material uncertainties. In Elastoplasticity the soil is assumed to behave elastically up to a certain stress limit after which combined elastic and plastic behaviour occurs.

For studying uncertainty in elastoplasticity Jeremic et al. (2007), based on a formulation in terms of probabilistic elastoplasticity, has shown interesting results, but the implementation seems arduous. In Karapiperis et al. (2016) and based on the principles of probabilistic elastoplasticity, the authors proposed a solution of stochastic elastoplastic boundary value problems with non-Gaussian parametric uncertainty. They illustrated their approach in a static shear beam example of elastic-perfectly plastic as well as isotropic hardening material. The work requires very sophisticated maths to solve the equations and presumably long computing time. Most of the other approaches that deal with uncertainties in soil modelling use random finite elements which consume tremendous computing time.

Fuzzy sets appeared in soil modelling by Klisinski (1988) who proposed to use them in the formulation of elastoplasticity. The main feature in his work is the use of a "fuzzy" yield surface, to study the cyclic loading behaviour of different materials. From the constitutive modelling point of view, it is closely related to many previous cyclic plasticity models. Instead of utilizing two or more yield or bounding surfaces the author introduced a general surface in the space spanned by the stress and a membership function, which allows to define different models within the same mathematical framework. Following the work of Klisinski (1988), Bao and Sture (2011) applied a kinematic-cyclic constitutive model based on the fuzzy-set concepts. They used the approach to deal with varying yield function and investigated model's capability of modelling soil dilatancy.

Wang et al. (2021) introduced the use of a plastic membership function varying from 0 to 1 to model continuous transitions inside and outside the initial yield surface. Within the fuzzy plasticity theory, the stress inside and outside the initial yield surface can produce plasticity. According to the authors absolute plasticity was not observed; only different degrees of plasticity existed.

The transition between the elastic and plastic states is sharp in the classical plasticity theory. To overcome the problem, and in order to capture nonlinear hardening behaviour and smooth transition from elastic to plastic state, Chunliang & Liangchi (2010) proposed a model of fuzzy plasticity based on the theory of fuzzy sets and Takagi-Sugeno inference. In the proposed model a set of surfaces, to partition the stress space with individual plastic modulus, is used. The model was applied on stainless steel and showed ability to handle many practical problems that cannot be adequately analysed by the conventional theory of plasticity.

In their contribution, Yang et al. (2015) worked on considering the effects of uncertainties in a classical elastoplastic analysis of structures. They were interested in how the uncertainties are related to the applied forces and the plastic material capacities, both of which are taken to lie within deterministic but bounded intervals. The interval approach requires knowledge of the upper and lower bounds of the uncertain data, and yet can provide fruitful preliminary sensitivity-type solutions to some response variable, prior to any comprehensive probabilistic and/or imprecise probabilistic analysis.

Zalewski et al. (2009) studied the influence of uncertainty in physical parameters in elasticity problems. They addressed the treatment of geometrical uncertainty in elasticity problems based on fuzzy boundary element method, and have shown interesting results. The uncertainty in the geometry was considered in terms of fuzzy quantities. The obtained results after analysis appear as fuzzy quantities as well.

In the proposed approaches, significant improvements were shown, particularly in the formulation of a “fuzzy” gradual yield function, however the uncertainty that can emerge from the material properties was not considered. There is significant influence of the material properties uncertainty on the mechanical behaviour.

Pramanik and Baidya (2022) tried to assess the reliability of elastic settlement of rectangular footing resting on both homogeneous and layered cohesionless soil strata using the fuzzy set theory coupled with the finite element method. The interest in their study lays in assuming the elastic modulus E_s of the soil as a fuzzy variable. The parameter E_s is considered as fuzzy variable maintaining some membership function, which is generally taken as either a triangular or trapezoidal type. The obtained outputs, elastic settlements, are also expressed as fuzzy variables and in this case, they maintain uncertainty as a result of the input parameter uncertainty.

Kruis and Stembek (2005) attempted to introduce the effect of uncertainty to the Chen model of plasticity used for concrete. The material parameters were considered in the form of fuzzy numbers. The authors proposed a fuzzification system for the material model and have shown through an illustrative example the applicability of the proposed approach. In another analysis Lopez-Caballero et al. (2010) adopted uniform probability distributions for the material properties K and G, and analyzed the distribution of elastoplastic stress-strain relations in triaxial tests. The authors performed 1000 sample Monte-Carlo computations to obtain valuable results.

The main objective of this study is to deal with parameter uncertainty and how to incorporate it in soil behaviour models using a fuzzy based approach. Interval arithmetic is of importance in this case, a fuzzy number being a superposition of α-level cuts (intervals). The main parameters of interest in the study are the elastic properties of the geomaterial K and G which are taken as uncertain based on fuzzy numbers. We first go through a summary on elastopasticity for geomaterials, with a particular focus on Mohr-Coulomb model. Secondly, we express the main features of the fuzzy-based approach followed by studied examples and discussions.

2 THEORY AND METHODS

2.1 *Elastoplasticity for soils*

The principles of elastoplasticity for soils rely generally on:

- Defining an elastoplastic constitutive relation between stress and strain,
- Defining a yield function,
- Defining a flow rule and
- Setting a hardening rule (Nordal et al., 2016)

The elastoplastic constitutive matrix is defined by:

$$d\boldsymbol{\sigma}=\boldsymbol{D}_{ep}d\boldsymbol{\varepsilon} \tag{1}$$

Where $d\boldsymbol{\sigma}=\{d\sigma_x, d\sigma_y, d\tau_{xy}\}$ is the stress increment vector, and $d\boldsymbol{\varepsilon}=\{d\varepsilon_x, d\varepsilon_y, d\gamma_{xy}\}$ the strain increment vector.

$$\boldsymbol{D}_{ep}=\left[\boldsymbol{D}-\frac{\boldsymbol{D}\left\{\frac{\partial Q}{\partial \boldsymbol{\sigma}}\right\}\left\{\frac{\partial F}{\partial \boldsymbol{\sigma}}\right\}^T\boldsymbol{D}}{\left[A+\left\{\frac{\partial F}{\partial \boldsymbol{\sigma}}\right\}^T\boldsymbol{D}\left\{\frac{\partial Q}{\partial \boldsymbol{\sigma}}\right\}\right]}\right] \tag{2}$$

Where, $\boldsymbol{D}$ is the elastic matrix, function of K and G, the bulk and shear modules of soil.

$\frac{\partial F}{\partial \boldsymbol{\sigma}}$ is the gradient to the yield surface F.
$\frac{\partial Q}{\partial \boldsymbol{\sigma}}$ is the gradient to the plastic potential Q.
A is the hardening parameter.

The elastic matrix is given by:

$$\boldsymbol{D}=\begin{bmatrix} K+\frac{4}{3}G & K-\frac{2}{3}G & 0 \\ K-\frac{2}{3}G & K+\frac{4}{3}G & 0 \\ 0 & 0 & G \end{bmatrix} \tag{3}$$

F is set for the yield function and Q the plastic potential.

2.2 Mohr-Coulomb model

The Mohr-Coulomb elastoplastic model is considered in this study, where the yield function is given by :

$$f = (\sigma'_1 + at) - N_f(\sigma'_3 + at) = 0 \qquad (5)$$

σ'_1 and σ'_3 are the major and minor effective stresses and at stands for attraction.

$$N_f = \frac{1+sin\varphi}{1-sin\varphi} \qquad (6)$$

For an equilibrium stress state below failure we use the mobilized friction angle ρ for which a degree of mobilization $f = \frac{tan\rho}{tan\varphi}$ measures how far is the stress state from failure.

$N_f = \frac{1+sin\rho}{1-sin\rho}$, where ρ is mobilized friction angle.

The plastic potential is given by:

$$Q = (\sigma'_1 + at) - N_Q(\sigma'_3 + at) \qquad (7)$$

$$N_Q = \frac{1+sin\psi}{1-sin\psi} \qquad (8)$$

where ψ is the dilatancy angle.

In plane strain condition, if $d\gamma_{xy} = 0$, we can reduce the elastic matrix $\boldsymbol{D}$ to:

$$\boldsymbol{D} = \begin{bmatrix} K + \frac{4}{3}G & K - \frac{2}{3}G \\ K - \frac{2}{3}G & K + \frac{4}{3}G \end{bmatrix} \qquad (9)$$

And $d\boldsymbol{\sigma} = \{d\sigma_x, d\sigma_y\}$, as well as $d\boldsymbol{\varepsilon} = \{d\varepsilon_x, d\varepsilon_y\}$

3 UNCERTAINTY IN ELASTIC PARAMETERS

Uncertainties in geotechnics are generally classified into two categories, “epistemic” related to lack of data or knowledge and “aleatory” related to natural randomness. Uncertainty in the elastic parameters of soil relates mainly to the epistemic category as it originates from the procedures and techniques for measuring the parameters. It is generally considered using probabilities. Parameter uncertainty can affect considerably the output of elastoplastic soil models as it propagates through the constitutive relations.

An example of uncertainty in the elastic modulus is shown in the results of a review by Selvadurai et al. (1979) where it was concluded that the undrained elastic modulus can be approximated by :

$$\frac{\delta}{\frac{p \times r}{E_u}} = [0.60, 0.75] \qquad (10)$$

Where:
δ : plate displacement
p : average stress on plate
r: plate radius
E_u: undrained elastic modulus

The formula is based on the screw plate test in a homogeneous, isotropic, and elastic medium. As we can see there is significant level of uncertainty in the undrained elastic modulus. In Cushing and Kulhawy (2001) many results on elastic soil uncertainty are shown, and the variability of soil elastic properties is preponderant. Karacan (2009) calculated and showed the variability and uncertainty in shear and young moduli for which calculations were based on compression and shear-wave sonic velocities. Polzl (2012) in a study of the correlations between calculated dynamic elastic moduli and measured or calculated static moduli, has shown how prevalent are uncertainties in the soil parameters. We propose to use fuzzy-based approaches for approximating the uncertainties in the soil properties of an elastoplastic model.

3.1 Reminder on intervals and fuzzy sets

Interval analysis was introduced by Moore (1966), it deals with quantities expressed as intervals which are common in engineering problems. Interval-valued approaches are convenient when dealing with uncertainty due to lack of data. In this study, the elastic parameters K and G will be considered as interval-based. A superposition of intervals (α-level cuts) leads to the construction of fuzzy sets for K and G.

A fuzzy set is a set of ordered pairs, $[x, \mu(x)]$, where a member x belongs to the set in a certain degree, termed membership grade $\mu(x)$. These ordered pairs collectively define a membership function that specifies a membership grade for each member (Zadeh, 1965). The triangular membership function, shown in Figure 1 is the most widely used membership function for representing uncertain soil parameters in geotechnical practice (Gong et al., 2014).

A triangular fuzzy number (figure 1) is denoted by $\mathrm{u} = \langle \mathrm{a}, \mathrm{b}, \mathrm{c} \rangle$, where $\mathrm{a} \le \mathrm{b} \le \mathrm{c}$, has α-level cuts:

$$[\mathrm{u}]_\alpha = [\mathrm{a} + \alpha(\mathrm{b} - \mathrm{a}), \mathrm{c} - \alpha(\mathrm{c} - \mathrm{b})], \alpha \in [0,1]$$

And membership function $\mu(x)$

$$\mu(x) = \begin{cases} \frac{x-a}{b-a} & if\ a \leq x \leq b \\ \frac{c-x}{c-b} & if\ b \leq x \leq c \\ 0 & otherwise \end{cases} \quad (10)$$

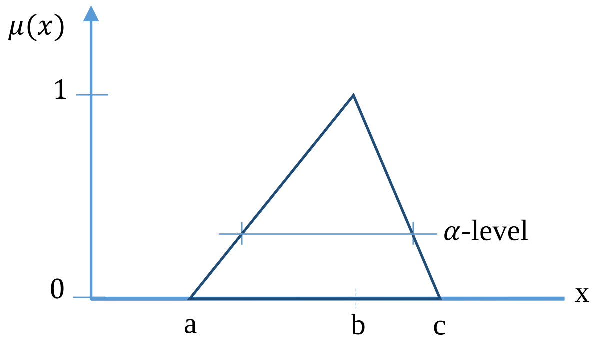

Figure 1. Fuzzy triangular number

Arithmetic operations on fuzzy numbers are based on interval arithmetic at α-level cuts.

3.2 Hartley-like measure of uncertainty

The Hartley-like measure of uncertainty was introduced by Klir (2011), it is derived from Hartley measure for graded possibilities π, and it applies in the same way to fuzzy sets.

$$UL(\pi) = \int_0^1 log_2\,[1 + \mu(\alpha_\pi)]d\alpha \quad (11)$$

$\mu(\alpha_\pi)$ is the Lebesgue measure, it represents the length of α-cut in this case.

In this study, we use the Hartley-like measure to estimate the output stress uncertainty and how it evolves in the elastoplastic process. The uncertainty of input elastic parameters remains constant while the output uncertainty will vary.

Using $\widetilde{K}$ and $\widetilde{G}$ for fuzzy bulk and shear modules the elastic matrix will be written in fuzzy terms as it follows:

$$\widetilde{\boldsymbol{D}} = \begin{bmatrix} \widetilde{K} + \frac{4}{3}\widetilde{G} & \widetilde{K} - \frac{2}{3}\widetilde{G} \\ \widetilde{K} - \frac{2}{3}\widetilde{G} & \widetilde{K} + \frac{4}{3}\widetilde{G} \end{bmatrix} \quad (12)$$

We can also use $\tilde{E}$, the elastic modulus, as a fuzzy number to obtain $\widetilde{K}$ and $\widetilde{G}$ using the relations:

$$\widetilde{G} = \frac{\tilde{E}}{2(1+\upsilon)} \quad (13)$$

$$\widetilde{K} = \frac{\tilde{E}}{3(1-2\upsilon)} \quad (14)$$

For the sake of representation, the fuzzy numbers are written in terms of intervals at a given α-level cut.

$\widetilde{K}$ is replaced by K_α^I and $\widetilde{G}$ by G_α^I. The index I is used to notice a given interval I of α-level cut. $\underline{K}(\alpha), \overline{K}(\alpha)$ are the lower and upper values of the interval K_α^I at α-level cut. The same representation is adopted for G_α^I with $\underline{G}(\alpha), \overline{G}(\alpha)$ as the lower and upper values of the interval at α-cut level.

Under triaxial loading conditions the material will follow an elastoplastic behaviour and the obtained stress $\boldsymbol{\sigma}$ at each step will be affected by the initial uncertainty of the soil elastic parameters.

4 CALIBRATION AND RESULTS

The model was calibrated using results of a triaxial compression from Xiang and Zi-Hang (2017) where the elastic modulus $E = 2300\ kPa$, the Poisson ration $= 0.4$, the cohesion of the soil $c = 21.43\ kPa$, the angle of internal friction $\phi = 17.13°$ and the unit weight $\gamma = 17.8\ kN/m^3$. In the simulation of the triaxial compression a confining pressure of 63 kPa was applied, and a downward displacement of 16 mm was driven on its top. Figure 2 shows the curve of the deviatoric stress (kPa) vs axial strain.

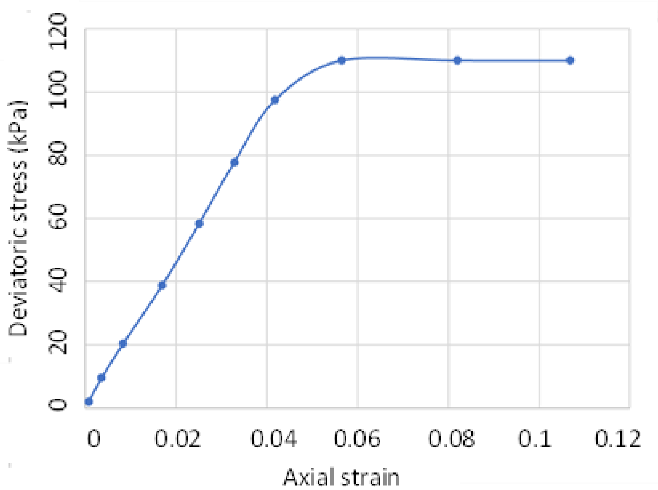

Figure 2. Stress-strain curve of numerical simulation – Triaxial compression (digitised from Xiang and Zi-Hang,2017)

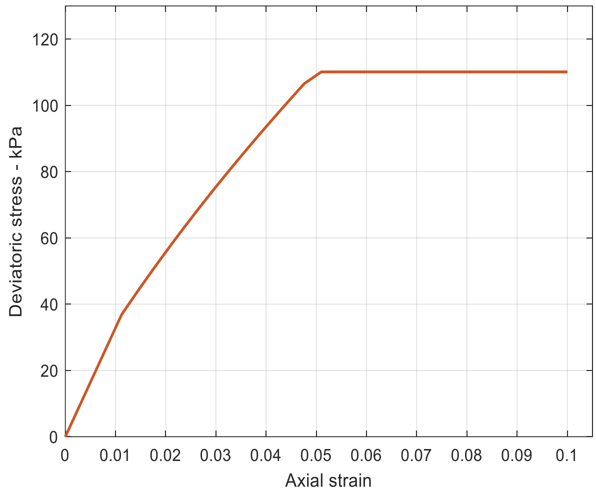

Figure 3. Simulation of triaxial test, deviatoric stress-axial strain ε_1 (present model)

4.1 Model with fuzzy elastic parameters

As shown in section 4, the soil elastic parameters (E, G, K) will be considered with uncertainty using fuzzy variables that can be constructed from histograms of laboratory results or by assigning approximate membership functions. The most common shape for the membership functions is the triangular distribution. For the sake of illustration, we consider uncertainty using a triangular shaped fuzzy variable for $\tilde{E}$. The fuzzy variables $\tilde{G}$ and $\tilde{K}$ are calculated using expressions (13) and (14), however due to space limitation they are not represented here.

We present in this section the results of a simulation of Mohr-Coulomb model under triaxial conditions. We use $\tilde{E}$ as a fuzzy variable where, $E_a = 2100\ kPa$, $E_b = 2300\ kPa$ and $E_c = 2400\ kPa$. And the obtained $\tilde{G}$ and $\tilde{K}$ using (13) and (14) were, respectively; $G_a = 750\ kPa$, $G_b = 821.43\ kPa$, $G_c = 857.14\ kPa$ and $K_a = 3500\ kPa$, $K_b = 3833.3\ kPa$, $K_c = 4000\ kPa$.

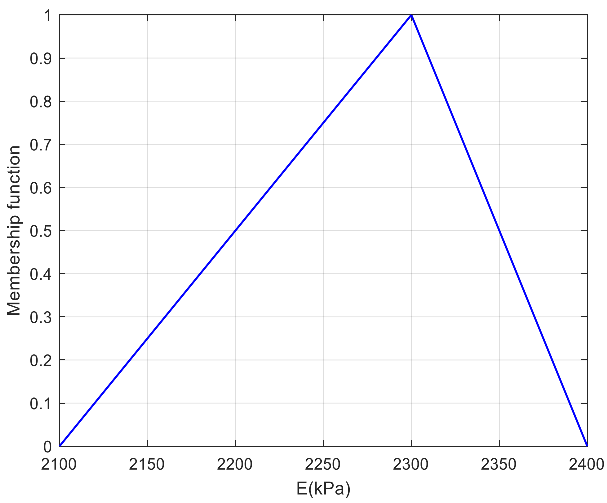

Figure 4. Fuzzy elastic modulus $\tilde{E}$

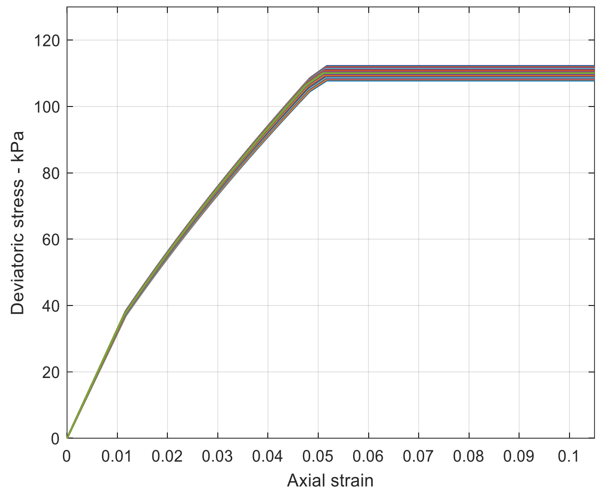

Figure 5 Fuzzy deviatoric stress vs axial strain ε_1

Figure 5 shows the deviatoric stress-strain evolution with the effect of elastic parameters uncertainty on the stress. Figure 6 depicts the stress profiles at some levels of strain ($\varepsilon_1 = 0.02\,; 0.04\,; 0.05$ and 0.06) where we can clearly see the evolution of the stress from one fuzzy configuration to the other. The triangular shape is maintained with the uncertainty increasing between $\varepsilon_1 = 0$ and $\varepsilon_1 = 0.05$ (axial strain) and a stable shape after that. The main variation in uncertainty is occurring in the elastic part, while in the plastic part it reached a stationary value.

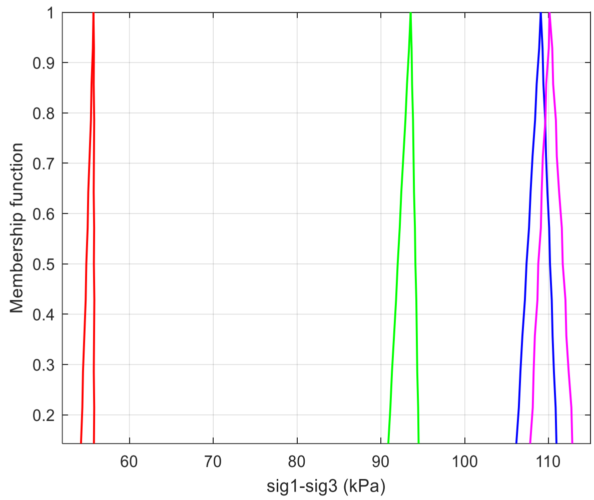

Figure 6. Fuzzy deviatoric stress $(\sigma_1 - \sigma_3)$ at axial strain $\varepsilon_1 = 0.02\,; 0.04\,; 0.05$ and 0.06

The Hartley-like uncertainty is an efficient tool for measuring uncertainty at different steps. It was measured for the evolving deviatoric stress as shown in figure 7 where we can see a rapid increase between $\varepsilon_1 = 0$ and $\varepsilon_1 = 0.011$ (axial strain) followed by a linear evolution of the uncertainty level from $\varepsilon_1 = 0.011$ to approximately $\varepsilon_1 = 0.05$ of axial strain, then a plateau of constant uncertainty is remarked until the end of the elastoplastic process.

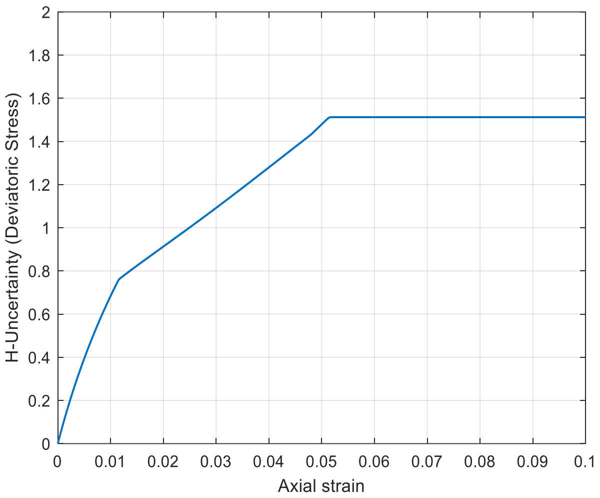

Figure 7. Evolution of Hartley-like uncertainty for the fuzzy deviatoric stress $(\sigma_1 - \sigma_3)$

5 CONCLUSIONS

In this study we tried to consider the effect of elastic parameters uncertainty on elastoplastic behaviour of clay. Mohr-Coulomb model in triaxial conditions was used to show that uncertainty evolves increasingly until a certain level of strain and then stabilizes for the rest of the process. Hartley-like measure of uncertainty is suitable as it helps following the evolution of stress-strain uncertainty with precision.

The shape of the fuzzy elastic parameters was reproduced in the stress-strain elastoplastic relation. The output stress was shown to maintain the triangular shape with different levels of uncertainty.

The studied example was used for illustration. Research on a large number of cases would help understanding more the consequences of considering parameter uncertainty in elastoplastic soil modelling.

6 REFERENCES

Bao, Y. , Sture, S. 2011. Numerical modelling of cyclic mobility based on fuzzy-set concepts in plasticity theory. *Computers and Geotechnics*, **38** (2011) 375-382.

Chunliang, Z. , Liangchi, Z. 2010. A general model of fuzzy plasticity. *Chinese journal of mechanical engineering* **23**, no. 4.

Cushing, A.G., Kulhawy, F.H. 2001. Undrained elastic be havior of drilled shaft foundations in cohesive soils. Conference: *15th International Conference on Soil Mechanics and Geotechnical Engineering*, Istanbul, Turkey.

Drucker, D.C., Prager, W. 1952. Soil mechanics and plas tic analysis or limit design, *Quart. Appl. Math*., **10**, 157–165.

Gong, W, L., Wang, S., Khoshnevisan, C.H., Juang, H., Huang, Zhang, J. 2014. Robust geotechnical design of earth slopes using fuzzy sets. *J Geotech Geoenviron Eng*. https://doi.org/10.1061/(ASCE)GT.1943-5606.0001196

Karacan, Ö. 2009. Elastic and shear moduli of coal measure rocks derived from basic well logs using fractal statistics and radial basis functions. *International Journal of Rock Mechanics and Mining Sciences*. **46**, Issue 8, 1281-1295

Karapiperis, K., Sett, K., Levent Kavvas, M., Jeremic, B. 2016. Fokker–Planck linearization for non-Gaussian stochastic elastoplastic finite elements. *Computer Methods in Applied Mechanics and Engineering*. **307**. 451-469

Klir, G.J. 2011. A note on the Hartley-like measure of uncertainty. *International Journal of General Systems*, **40**:02, 217-229, DOI: 10.1080/03081079.2010.526757.

Klisinski, M. 1988. Plasticity theory based on fuzzy sets. *Journal of Engineering Mechanics*. Vol. **114**, Issue 4.

Kruis J., Stemberk, P. 2005. Fuzzyfication of Chen model of plasticity of concrete. *VIII International Conference on Computational Plasticity COMPLAS VIII*. I E. O˜nate and D.R.J. Owen (eds) CIMNE, Barcelona, Spain.

Lopez-caballero F., Modaressi-Farahmand Razavi, A. 2010. Uncertainty and sensitivity analysis of laboratory test simulations using an elastoplastic model. *7th European Conference on Numerical Methods in Geotechnical Engineering*, Jun 2010, Trondheim, Norway.

Moore, R.E. 1966. *Interval analysis*. Prentice-Hall, Englewood Cliff.

Nordal S., Grimstad, G., Benz, T. 2016. *Soil modelling*. PhD course, Geotechnical Group. NTNU, Trondheim, Norway.

Polzl, H. 2012. *Correlations and empirical relations between static and dynamic elastic ground parameters in shallow geotechnical site investigations*. MSc. Thesis, A-8700, Montanuniversität Leoben, Germany.

Pramanik, R., Kumar Baidya, D. 2022. Influence of spatial variability of soil elastic modulus on elastic settlement of rectangular footing under fuzzy uncertainty. *Innovative Infrastructure Solutions* **7**:224 https://doi.org/10.1007/s41062-022-00819-7

Selvadurai, A.P.S., Nicholas, T.J. 1979. A theoretical assessment of the screw plate test. *Third International Conference on Numerical Methods in Geomechanics*, Aachen, Germany, Vol. 3.

Wang X., Jin, L., Fu, M. 2021. Research on fuzzy plastic constitutive model based on membership function. *Mathematical Problems in Engineering* Volume **2021**, Article ID 9901948. https://doi.org/10.1155/2021/9901948.

Xiang, X., Zi-Hang, D. 2017. Numerical implementation of a modified Mohr–Coulomb model and its application in slope stability analysis. *J. Mod. Transport*. **25**, 40–51. https://doi.org/10.1007/s40534-017-0123-0

Yang C., Tangaramvong, S., Gao, W., Tin-Loi, F. 2015. Inter val elastoplastic analysis of structures. *Computers and Structures* **151** (2015) 1–10.

Zadeh, L. 1965. Fuzzy sets. *Information Control* **8**(3):338–353

Zalewski, B.F., Mullen, R.L., Muhanna, R.L. 2009. Fuzzy boundary element method for geometric uncertainty in elasticity problem. *SAE Int. J. Mater. Manuf*. **2(1)**:310-316.