Ramification of remanufacturing in a sustainable three-echelon closed-loop supply chain management for returnable products

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Abstract

The closed-loop supply chain management (CLSCM) is an attractive research field for the corporate and academic worlds; however, closing the loop is not a simple task. Reverse logistics activities increase management complexities and uncertainties by establishing multi-fold collection and return management processes. Unlike traditional supply chain management, where managers deal with only stochastic demand, in closed-loop supply chain management, they deal with both stochastic demand and returns, which increases the cumulative uncertainty in the system. Firms usually use disposable packaging, and demand uncertainties also increase the negative environmental implications of logistics activities. This study aims to investigate optimal remanufacturing strategy and reusable packaging capacity under stochastic demand and return rate for single and multi-retailer closed-loop supply chain models. The results show that a hybrid policy is an optimal option for both single and multi-retailer cases; however, the rate of remanufacturing increases for multiple-retailers. Furthermore, remanufacturing cost, manufacturing cost, and ordering cost of retailers are the principal drivers of hybrid supply chain management. The results further suggest that supply chain managers should reduce manufacturing and remanufacturing costs because they play a central role in deciding the optimal remanufacturing rate. Increasing the remanufacturing rate increases ordering quantities and reduces setup and ordering costs in the system. Thus the remanufacturing is a relatively inexpensive policy for supply chains with higher setup and ordering costs. Numerical examples, sensitivity analysis, and comparative study show the robustness and validity of the proposed model.

Keywords: Sustainability; Supply chain management; Remanufacturing, Reverse logistics; Returnable transport items.

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1 Introduction

The traditional business practices of landfilling products after their useful life are not considered sustainable any more because of two main reasons. Firstly, they increase natural resource depletion, thus, increasing the cost to extract more raw materials to fulfill the demand of consumers. Secondly, they destroy the natural ecosystem by contaminating soil, water and air. Therefore, worldwide business legislations are now forcing firms to explore ways of reducing resource consumption and waste generation (Moshtagh and Taleizadeh, 2017). For instance, the Extended Producer Responsibility (EPR) that extends the manufacturer's responsibilities to the post-consumer stage of the product's lifecycle (Ullah and Sarkar, 2020), mandated recycling, minimum recycled content standards, energy efficiency standards, disposal bans and restrictions, advance recycling fee (ARF), advance disposal fee (ADF), virgin material taxes/subsidies and deposit/refund schemes are some of the relevant laws which push manufacturers towards industrial sustainability and reduction of solid waste generation (Habib et al., 2019; Ullah and Sarkar, 2018). Fortunately, we have a solution to both of the above-mentioned problems in the form of recovery-and-reuse policy that can reduce both natural resource consumption and solid waste generation. The recovery-and-reuse policy stands on 6Rs that are reduce, reuse, recycle, redesign, remanufacture and refurbish. This research work takes reduce, remanufacture and reuse into considerations to decrease both natural resource consumption and solid waste generation.

Remanufacturing, in which used products are recovered and turned again into useful products, is a profitable way of reducing natural resource depletion and waste generation (Liu et al., 2020). It intends to revive residual business value from used products through either component replacement or reprocessing and gives them a new life. Traditionally, remanufacturing is considered an economical option compared to manufacturing new items (Sundin and Bras, 2005) because it allows getting a product to meets the demand of the market for a lower price point than a brand new product (Iqbal and Sarkar, 2019). However, more recently, it gains importance because of its environmental benefits compared to the economic ones. Now, remanufacturing is considered a highly sustainable practice because of its higher absolute environmental impact in most industries that depend on non-renewable resources. Because of its economic benefits, remanufacturing provides new means for sustaining an edge in a hyper-competitive market environment. Therefore, from a commercial perspective, 100% remanufacturing may seem more desirable because of low remanufacturing costs. However, in the broader picture, with other operational costs like recovery, management and logistics taken into account, the results vary significantly (Klumpp et al., 2019; Lu and Liu, 2011; Sarkar et al., 2017a). Therefore, the concept of hybrid manufacturing-remanufacturing systems was introduced. Through which partial demand is satisfied with remanufacturing while the remainder is fulfilled from manufacturing.

Additionally, remanufacturing is enabled by the collection and reprocessing of used products that lead to additional reverse logistics and management activities in the supply chain. Because transport packaging is one of the largest contributors to manufacturing waste, increasing transportation raises solid waste generation from supply chain management. Thus, affecting the overall sustainability and economics of remanufacturing simultaneously. In this regard, many industries have considered using reusable transport packaging rather than disposable packaging. RTIs, known as Returnable Transport Items, are reusable secondary transport packages such as pallets, crates, rail-cars and containers and are vital to the sustainability of logistics activities. Literature suggests that the reuse of transportation packaging reduces both energy consumption and solid waste generation from transportation; and cut down CO_2 discharge by up to 16% (Hekkert et al., 2000; Piecyk and McKinnon, 2010). However, the use of returnable transport items needs adequate coordination with supply chain operations to achieve the benefits to the full extent. Because the mismanagement of RTIs produces additional complexities for high volume and high-value packaging materials and increase the management and transportation costs of the system. The severity of these problems is further increased in Closed-Loop Supply Chain (CLSC) because of the complex recovery system. Hence, it is essential to regulate the management of RTIs for a smooth circulation of materials in the multi-echelon supply chain environment. To answer this problem, we investigate a three-echelon supply chain model with a third party logistics (3PL) that control all transport activities for the whole supply chain.

However, all these economic and environmental benefits of remanufacturing, reuse and closing the loop are not free of cost. They all increase the complexity and uncertainty of the operations that arise from Reverse Logistics (RL) activities. These activities generate undesired in-process costs in the remanufacturing system if not managed properly Wang et al. (2019). The randomness in return rate and remanufacturing cost makes it challenging for production firms to manage the process smoothly. Asghar et al. (2019) and Asghar and Kim (2020) showed that stochastic failure and repair rate increase the complexity, costs and production time for manufacturing process. Consequently, this adds to the inherent complexity of CLSC with the additional costs imposed by recovery operations. Especially, when demand is stochastic, the uncertainty makes closed-loop logistics operations even more cumbersome (Hota et al., 2020). The motivation for this paper stems from theses challenges of hybrid stochastic manufacturing-remanufacturing systems and RL activities. The aim of this study is to develop optimal remanufacturing strategy for a manufacturing-remanufacturing system that uses RTIs for product transportation. Furthermore, this study also aims to investigate the key supply chain drivers of the CLSCM under the stochastic environment. A three-echelon green supply chain model composed of retailers, a manufacturer, and a 3PL is developed and studied under both single retailer and multi-retailer settings. The optimal remanufacturing rate, RTI capacity, and retailer order size are considered to improve resource utilization and to reduce environmental consequences of manufacturing and logistics activities.

Finally, rest of the paper is arranged as follows: Section 2 presents an overview of the associated literature; Section 3 gives problem definition; further Section 4 derives mathematical model of the three-echelon CLSC; Section

5 investigates the developed mathematical model through numerical examples, sensitivity analysis, and comparative analysis of different model parameters. Finally, Sections 6 deduces the paper and recommends future extensions of the developed model.

2 Literature review

This study deals with three different research streams of CLSC literature including hybrid-stochastic manufacturing-remanufacturing systems; returnable transport items and stochastic demand of products. This section presents a comprehensive rundown of existing literature in well-reputed databases such as Web of Science, Scopus, ScienceDirect, Emerald, Wiley, Google Scholar and SAGE full-text collection. The main keyword is supply chain management, then the results are limited for each relevant research stream. The first stream of research looks into the economic performance of hybrid manufacturing-remanufacturing systems under uncertain environment. Additionally, the hybrid manufacturing-remanufacturing is enabled by the collection and reprocessing of used products which lead to additional logistics and management activities. Moreover, the reduction of solid waste is the ultimate goal of sustainable manufacturing practices to avoid many environmental consequences and costs before they occur. Therefore, the second stream of this research looks into RTI for hybrid sustainable CLSC transport activities. To make the CLSC management problem more realistic, the third stream introduces the influence of random demand and return rate for the hybrid system.

2.1 Hybrid systems

Worldwide business legislation are now forcing manufacturers to adopt the sustainable strategies for products which help to save natural resources Moshtagh and Taleizadeh (2017). Recovery and recycling of products after their useful life are ultimate approaches towards sustainability which seems promising for manufacturers. The CLSC management is referred to the efforts or activities of recovering assets that would be wasted away otherwise (Mishra et al., 2020a; Nasr, 2019). For instance, industrial products like computers, automotive parts, metals, cranes, fork-lifts, medical equipment, tires, electrical equipment and toner cartridges, etc. are examples of remanufacturable products (Lund and Skeels, 1983). Research in CLSC originates back to 1960's, when Schrady (1967) introduced the concept of remanufacturing within inventory management for the first time. The study examined the famous economic order quantity model (EOQ) with repair and named it (R, I) policy. The model considers instantaneous and infinite manufacturing and repair rate. This work was further extended to a hybrid manufacturing-remanufacturing system by Richter (1997). Mabini et al. (1992) examined a demand-dependent return rate. Further studies for the hybrid strategy with several production and remanufacturing cycles were conducted by Richter and Dobos (1999); Teunter (1998). Koh et al. (2002) examine the optimum joint production policy for manufactured and recoverable products. These studies investigate a non-instantaneous manufacturing and repair system for constant demand rate. Dobos and Richter (2004) extends the work to multiple manufacturing and repair batches per cycle. A hybrid system is analyzed with quality consideration in Dobos and Richter (2006), which suggests mixed strategy as more beneficial than the pure manufacturing/remanufacturing ones. El Saadany and Jaber (2010) suggested that a hybrid policy is an optimal strategy when recovery rate is price and minimum acceptable quality dependent.

Recently, stochastic optimal control policy for a deteriorating hybrid manufacturing-remanufacturing system is developed by Ouaret et al. (2018). The study considers an unreliable deteriorating machine to optimize production and replacement policies simultaneously. Optimal production scheduling for hybrid manufacturing-remanufacturing policy has been investigated by Polotski et al. (2015). Also, Moshtagh and Taleizadeh (2017) investigated hybrid CLSC management under stochastic environment and shortages. The study considers variable demand rate for manufactured and recovered products with quality dependent return rate. A hybrid CLSC under consignment stock policy is examined in Hariga et al. (2017). The study collectively optimize the batch size and sequence for manufacturing-repairing products by a mixed integer non-linear programming model. Moreover, Sarkar et al. (2017b) studied environmental aspects in hybrid systems and found that collective optimization of both environmental and financial cost promotes less remanufacturing in mixed strategies. Dhaiban et al. (2018) studied optimal policies for hybrid system where the manufacturing rate is firmly related to disposal rate and inversely related to remanufacturing rate. Most recently, Assid et al. (2019) developed production policy considering unreliability for hybrid system using simulation based optimization approach.

2.2 Closed-loop logistics and RTIs

In present time, the closed-loop supply chain management is catching vigorous attention in the business and research society due to the high costs of product waste disposal and strict environmental laws imposed by legislation authorities (Kam et al., 2018; Menderes et al., 2017). A remanufacturing environment establishes as a closed-loop system, combining conventional forward logistics flows with reversed channel logistics. This CLSC environment enables the supply of materials/products in the forward flow from manufacturer to a customer and back from customer to the manufacturer known as reverse flow. The existing research in CLSC management, overlooked some critical aspects of logistics operations, for example, transportation packages cost, transport vehicle capacities, transport modes and logistics management costs. Transport packaging is the biggest contributor to manufacturing waste by affecting transport costs and sustainability simultaneously. In this regard, many industries have considered using reusable transport packaging rather than disposable packaging. RTIs, known as returnable transport items are a big consideration for transport activities in sustainable supply chain management.

Secondary transport packaging like RTI subdues the transport costs and adverse environmental impacts in a supply chain, as the RTI reduces the use of raw material for packaging, Bottani et al. (2015). Some pioneering studies of RTI inventory management include the work of (Buchanan and Abad, 1998; Goh and Varaprasad, 1986; Kelle and Silver, 1989; Sarkar et al., 2019). The research focus in this context is mainly about the deployment and replacement quantities of RTIs, shipment and replacement schedule. For instance, Witt (2000) referred that the required number of RTIs is directly associated with the recollection rate of RTI in use. The growing emphasizes of consumers on environmental and ethical dimensions of products suggest a regime change in marketing ethics and labeled packaging of products. The preference analysis for green packaging also asserts significant insights towards change for transport packaging, Rokka and Uusitalo (2008). The lead time study for RTI inventory control is discussed by Silva et al. (2013). The reduction effect of return lead time of RTIs leads to positive economic and environmental impacts collectively. The negative impacts of secondary transport items have been discussed by Accorsi et al. (2014). The study reveals that use of RTIs increases the system costs if not managed appropriately. However, the use of RTI reduces negative environmental impacts. Furthermore, the profitability of RL is studied by many authors and the literature suggests a variety of parameters that affect the performance of CLSC (Mishra et al., 2020b; Sarkar et al., 2018; Ullah et al., 2019a). These parameters include transport packaging costs, quality of used products, remanufacturing cost, remanufacturing rate and management costs etc.. However, the impact of transport packaging cost and transport management activities on remanufacturing cost for a multi-echelon supply chain management has been largely overlooked by the literature.

In the CLSC system, smart planning/management for production quantities, inventory, transport, routing and shipment schedule are required to reduce carbon discharge while gaining maximum operational efficiency. To provide insight into this context, Zhang et al. (2018) discussed RL production-routing model with stochastic demand and emissions control. Recently, a RL supply chain model with diversified transportation fleets and remanufacturing is discussed by Shuang et al. (2019). The study investigates the logistics activities under cap-and-trade policy to optimize production, inventory and delivery quantities. To establish a better understanding of inter-dependencies among the RTIs management and production-distribution planning of products in a green CLSC, we investigate the logistical costs associated with RTIs in the hybrid-stochastic manufacturing-remanufacturing system.

2.3 Demand uncertainties in CLSC network

Quantification of influence of demand uncertainty on the production process and making the hybrid manufacturing-remanufacturing system applicable to the real-life production situation is the ultimate challenge of CLSC network design, (Habib et al., 2020; Niakan et al., 2016). Returned product quantity, quality and time, etc. are inherent random parameters of all closed-loop logistics system (İpekçi et al., 2018; Liao et al., 2018; Menderes et al., 2019). Among other uncertain aspects of a CLSC, demand rate is the most challenging random factor, as demand rate for the remanufactured products may not be thoroughly coordinated with return rate of used products. Based on the customer demand rates for different products, the appropriate quantity of manufactured/remanufactured products should be available in every stage of production and logistics to avoid shortages. Hence, it is crucial to examine the randomness of demand rate to deal with the complication of the supply-demand unbalance in various stages of the CLSC system. In this context, a capacitated multi-product reverses logistics network model is optimized by Salema et al. (2007), regarding the uncertainties related with the demand and return rate of used/collected products.

Likewise, Amin and Zhang (2012) developed MINLP models to examine the influence of demand and return rate uncertainties in CLSC system. An optimal approach, for hybrid manufacturing-remanufacturing, is examined in Guo and Ya (2015). In this study, the demand rate is constant and larger than the return rate. Moreover, both the buy-back price and remanufacturing cost depends on the quality of collected products. Furthermore, a multi-period recovery network optimization model with buy-back offers is proposed by (Dutta et al., 2016). The recovering rate of end of life/end of use (EOL/EOU) products is improved by assuming uncertain demand and capacity. Liao and Deng Liao and Deng (2018) developed an environmentally sustainable EOQ model with the uncertainty of market demand and quality acquisition. This study also investigated pure and hybrid remanufacturing strategies under stochastic distributions. To address the stochastic nature of product return, Fathi et al. (2015) investigated an analytical queuing model for a hybrid manufacturing-remufacturing system. With the fluctuations in the time value of money, inflation, and time-dependent holding costs, the economic aspects of production policies result in major restrictions for manufacturers. In this regard, work of Malik and Sarkar (2020) and Ahmad et al. (2018) provide better understanding of time-varying parameters in production policies. Environmentally conscious production policies are widely discussed in Kuo et al. (2006) and with flexible production policies in Joseph and Sridharan (2011).

Recently, product recovery management in a CLSC system is examined in Liao et al. (2019). The study states that it is more practical to investigate manufacturing, disposal and remanufacturing decisions under uncertainties of demand and return rate than considering them constant factors. To extend the hybrid manufacturing-remanufacturing model to a higher practical direction where product's demand is uncertain, we develop a green closed-loop logistics system by assuming market demand rate as a stochastic variable which follows a continuous distribution. In closed-loop logistics, it is intrinsically difficult to determine the return amount of used-products. Therefore, handling uncertainty regarding demand and return rate in a CLSC becomes even more challenging. The return rate of EOL/EOU products is generally considered comparative to the demand rate Moshtagh and Taleizadeh (2017) or quality dependent Giri and Sharma (2015). Whereas, in practice, the production firms have to collect used products from the market through additional recovery efforts like offering discounts, return coupons or additional incentives etc.. To address the real aspect of used product recovery in this study, the returned product amount is considered influenced by the recovery efforts and the return rate is considered a decision variable. Despite these tremendous efforts, there is paucity of research that have considered stochastic demand and return rate with transportation and sustainability in hybrid manufacturing-remanufacturing systems. The only research that consider transportation, emissions, and RTIs in hybrid systems is done by Sarkar et al. (2017b); however,

Table 1: Contribution of different authors

Author(s)	Demand	Return rate	RTI Packaging	Transportation	Sustainability
Rahmani and Yavari (2019)	variable		no	no	yes
Onggo et al. (2019)	stochastic		no	yes	no
Heydari et al. (2018)	deterministic	deterministic	no	no	no
Dou et al. (2019)	deterministic	variable	no	no	yes
Guo et al. (2019)	deterministic	variable	no	no	no
Heydari and Ghasemi (2018)	deterministic	variable	no	no	no
Liu et al. (2018)	variable	deterministic	no	no	no
Liu et al. (2019)	stochastic	stochastic	no	no	no
Cui et al. (2017)	uncertain	deterministic	no	yes	no
Dai and Zheng (2015)	uncertain	uncertain	no	yes	no
Almaraj and Trafalis (2019)	uncertain	uncertain	no	yes	no
Hassanpour et al. (2019)	uncertain	variable	no	yes	no
Vahdani and Ahmadzadeh (2019)	uncertain	uncertain	no	yes	no
Ahmadi and Amin (2019)	stochastic	uncertain	no	yes	no
Cobb (2016)	deterministic	random	yes	no	no
Iassinovskaia et al. (2017)	deterministic	deterministic	yes	yes	no
Wang and Zhao (2018)	variable	deterministic	yes	no	no
Sarkar et al. (2017b)	deterministic	deterministic	yes	yes	yes
Liao and Li (2020)	stochastic		no	yes	yes
Zhang and Chen (2020)	stochastic	exogenous	no	no	no
Alegoz et al. (2020)	uncertain	uncertain	no	yes	yes
Tolooie et al. (2020)	stochastic		no	yes	no
Mohammadi (2020)	stochastic		no	yes	no
This paper	stochastic	stochastic	yes	yes	yes

they consider deterministic demand and return rate, whereas this paper consider stochastic demand and return rate. Furthermore, their focus is on the relationship of RTIs and disposable package while this research focus on transportation and its impacts on remanufacturing rate. A brief comparison of literature is given in Table 1

3 Problem statement, notation, and assumptions

This sections defines the proposed problem, related notation, and assumptions for mathematical model.

3.1 Problem statement

This study develops a three-echelon CLSC model which consists of a single manufacturer, a 3PL, and multiple retailers. To make the CLSC management problem more realistic, the study examines the influence of random demand and return rate for the hybrid system. Additionally, the hybrid manufacturing-remanufacturing is enabled by the collection and reprocessing of used products, which lead to additional logistics and management activities in the supply chain. Therefore, a 3PL is hired for all forward and backward logistic activities in the hybrid supply chain. The manufacturer satisfy stochastic market demand of k retailers through a 3PL. The 3PL owns a collection setup in a remote area and provide logistics services to all supply chain members. Moreover, the reduction of waste products is the ultimate goal of sustainable manufacturing practices to avoid many environmental consequences and costs before they occur. Therefore, the proposed study considers RTI for hybrid sustainable CLSC transport activities and the 3PL transport goods using RTIs and delivers finished products from manufacturer to k retailers.

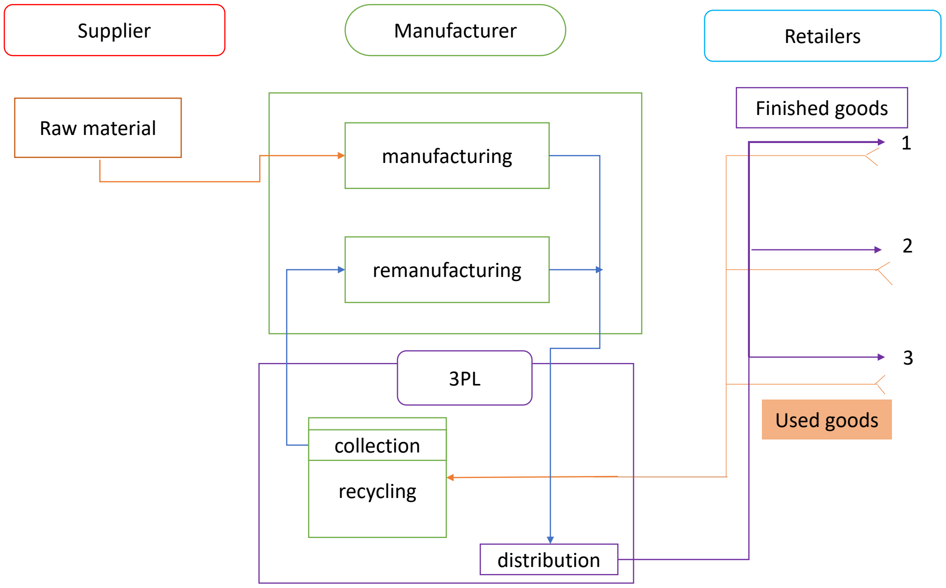

Figure 1: Logistics layout of three-echelon hybrid CLSC

The 3PL also manages reverses logistic operations from it collection center, collect the used products from market and transfer them back to the manufacturer. The manufacturer use collected products for remanufacturing along with producing new products. The 3PL charges collection and recycling cost to manufacturer to provide recovered products from the market. The whole scenario of Hybrid CLSC layout is illustrated in Fig 1. As, the demand of supply chain is considered stochastic which follows a uniform distribution over the time period T, the collection rate is considered as a decision variable for the 3PL. The study investigates the economic benefits of hybrid-stochastic manufacturing-remanufacturing under transport packaging choices.

3.2 Notation

The parameters used to develop the mathematical model are given below;

3.3 Assumptions

The study consider following given assumptions for derivation of CLSC mathematical model;

- The study assumes cooperation and information sharing, as a common interest of all supply chain players in the proposed integrated system. (Sarkar et al., 2017b). The market demand is stochastic and ensues a probability distribution over time period T, where manufacturing rate is assumed constant and larger than demand rate, $((\theta + 1 - \theta) > x = \sum_{j=1}^{k} x_j)$.

- The manufacturer produces two classes of product (new from fresh raw material and remanufactured from collected/used items) with the same finished quality q_f, where $(0 < q_f < 1)$. Both types of products are

Decision variables

θ	manufacturer's remanufacturing rate of used product
q_j	retailer's replenishment lot size, where $j = \{1, 2, ..., k\}$
τ	container capacity (units)

Model parameters

T	replenishment cycle time (year)
q_f	final quality of end products
q_u	minimum adequate quality of the collected items

Buyer's parameters

k	number of buyers
x_j	demand of j^{th} buyer, which is uniform over the period $[0, T]$ (units/unit time)
$f(x_j)$	probability density function for j^{th} retailer demand x_j
h_j	finished product holding cost at j^{th} buyer ($/unit/unit time)
O_j	ordering cost for j^{th} buyer ($/order)
l_j	lead time of j^{th} buyer, i.e. delivery time between buyer j and $j + 1$

manufacturer's parameters

O_m	ordering cost for 3PL ($/order)
S_m	setup cost ($/setup)
C_m	manufacturing cost per unit ($/product)
P_m	production rate (units/unit time)
C_{rm}	remanufacturing cost per unit for manufacturer ($/product)
h_f	finished product holding cost for manufacturer ($/unit/unit time)
C_q	quality improvement cost ($/product)
h_u	holding cost for used products ($/unit/unit time)
C_g	goodwill lost cost ($/product)
h_r	holding cost for raw materials ($/product)

3PL center's parameters

S_c	3PL's collection center setup cost for EOL/EOU products ($/setup)
γ	3PL's Investment for collection of EOL/EOU items/products in dollars ($)
C_{rc}	recycling cost per unit for collected products($/product)
h_c	3PL's cost for holding collected/used items ($/unit/unit time)
t_c	truck capacity (number of products)
C_t	cost of transportation per RTI per unit travelled distance ($/container/unit distance)
C_τ	container management cost including repair ($/unit capacity of RTI)
s	scaling parameter
h_R	holding cost for RTI ($/unit/unit time)
l_{jm}	distance (in kilometres) between j^{th} buyer and the manufacturer (kilometers)
l_{jc}	distance (in kilometres) between j^{th} buyer and collection center (kilometers)
l_{cm}	distance (in kilometres) between collection center and the manufacturer (kilometers)

Carbon emission's parameters

C_{ec}	carbon tax per unit of emissions ($/unit of discharged CO_2)
e_m	carbon discharge per manufactured item (kg/product)
g_m	fuel requirement of vehicle per unit distance (gallon/unit distance)
e_t	carbon discharge per gallon of fuel consumption (kg/gallon)

offered in market with similar selling prices.

- The transport of all collected products, finished products and maintenance (cleaning/repairing) of transport vehicles are the liabilities of 3PL. The 3PL owns a collection center which collects End of life/end of use (EOL/EOU) products from the customer, where collection rate θ, a fraction of demand, is considered a decision variable.

- The remanufacturing rate is assumed equal to collection rate θ of used products. Further, it is considered that collected fragment of demand θ, undergoes inspection and initial recycling processes at 3PL collection center, and only items with quality ($\geq q_u$) are purchased and recycled, where ($0 < q_u < q_f$).

- The average cost of recycling at 3PL collection center is assumed lower than the remanufacturing cost at manufacturer which is lower than the manufacturing cost, as given ($C_m > C_{rm} > C_{rc} > 0$) $\forall$ q_u (Maiti and Giri, 2015). After the completion of production, deliveries are made to k buyers where j^{th} buyer receives one delivery per production cycle of the manufacturer Sarkar et al. (2017b).

4 Mathematical model

This study considers a hybrid (manufacturing-remanufacturing) stochastic system, in which the manufacturer produces ($Q = \sum_{j=1}^{k} q_j$) products to fulfill stochastic market demand of k buyers. The demand distribution of all buyers is considered homogeneous, so that the total demand of system becomes ($x = \sum_{j=1}^{k} x_j$). The 3PL owns a collection setup, located at a distant market. The collection center recover EOL/EOU products with a collection rate θ, a fraction of demand, from widely diverse customers and bear associated transportation costs. The collected products undergo initial inspection and recycling for an acceptable quality range ($\geq q_u$), which are then shifted to production center. These products are then upgraded to quality q_f by remanufacturing process. The manufacturer produces two types of products (remanufactured and new products) with supposedly same finished quality q_f which incurs different production costs. Therefore, production costs for manufacturer consist of manufacturing, and remanufacturing costs. The 3PL incurs extra collection and recycling costs along with transportation costs. The proposed CLSC management system decides optimal transport packaging capacity and remanufacturing rate and minimize expected cost of the supply chain. Based on the assumptions stated earlier, this study formulates an optimization model for a sustainable CLSC network, which consists of buyer's cost, manufacturer's cost, and 3PL's costs. The whole CLSC scenario is depicted in Figure 2.

4.1 Buyer's costs

In the proposed model, each buyer places an order to the manufacturer, the manufacturer deliver those products to the buyer's place through 3PL, therefore, buyer objective function consists of ordering cost, expected inventory holding cost, and shortage costs. These are as given below:

4.1.1 Ordering cost (C_b^o)

This study considers multiple buyers that order ($q_j = x_j T$), where ($j = 1, 2, 3...., k$) quantities per cycle. As the j^{th} buyer place one order per cycle, therefore, the time weighted ordering cost becomes $\frac{O_j}{T}$.

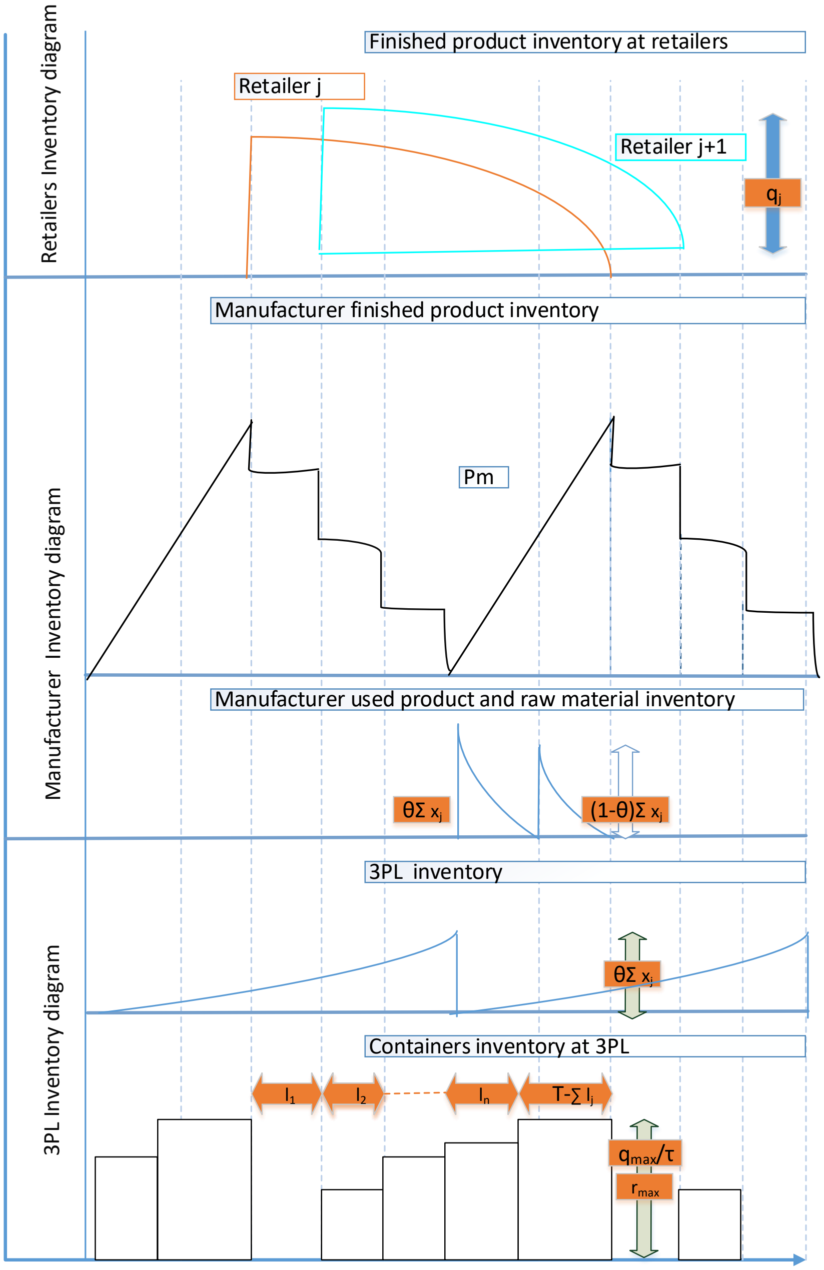

Figure 2: Inventory-logistics flow of the proposed stochastic-hybrid manufacturing-remanufacturing system.

4.1.2 Inventory holding cost (C_b^h)

As the demand is stochastic over the period $[0, T]$, two cases for demand rate arise:

Case 1 when no shortages occur, as ($q_j \geq x_j$)

The inventory holding cost for q_j products, over a period of T years is calculated as follows. The determining differential equation for j^{th} buyer's instantaneous inventory $q_j(t)$ is,

$$q_j(t) = \int \left(-\frac{x_j}{T} \right) dt. \qquad (0 < t < T) \tag{1}$$

Using the initial condition at $t = 0,\ q_j(0) = q_j$, the solution of Equation (1) is,

$$q_j(t) = \left(q_j - \frac{t x_j}{T} \right). \qquad (0 < t < T)$$

As no shortages are considered, so $q_j(0) = q_j$, implies that time weighted average inventory of j^{th} buyer is;

$$\int_0^T \left(q_j - \frac{t x_j}{T} \right) dt.$$

Now, the expected holding cost for finished products for j^{th} buyer inventory system is

$$h_j \int_0^{q_j} \left(T q_j - \frac{T x_j}{2} \right) f\left(x_j\right) dx_j.$$

Finally, the expected total inventory holding cost for k buyers is,

$$\sum_{j=1}^{k} \frac{h_j}{T} \int_0^{q_j} \left(T q_j - \frac{T x_j}{2} \right) f\left(x_j\right) dx_j. \tag{2}$$

Case 2 when shortages occur as ($q_j < x_j$)

$$q_j(t) = \int \left(-\frac{x_j}{T} \right) dt. \qquad (0 < t < t_r) \tag{3}$$

With the initial condition at $t = 0, \quad q_j(0) = q_j$, the solution of (1) is,

$$q_j(t) = \left(q_j - \frac{t x_j}{T} \right). \qquad (0 < t < t_r)$$

As at $q_j(0) = q_j$, implies that time weighted average inventory of j^{th} buyer is;

$$\int_0^{t_r} \left(q_j - \frac{t x_j}{T} \right) dt$$

Where $t_r = \frac{q_j T}{x_j}$, now, the expected holding cost of finished products inventory for k buyers is

$$\sum_{j=1}^{k} h_j \int_{q_j}^{\infty} \left(\frac{q_j^2}{x_j} - \frac{q_j^2}{2T x_j} \right) f\left(x_j\right)\, dx_j.$$

Now, total expected inventory from both cases become,

$$\sum_{j=1}^{k} h_j \int_0^{q_j} \left(T q_j - \frac{T x_j}{2} \right) f\left(x_j\right)\, dx_j + \sum_{j=1}^{k} h_j \int_{q_j}^{\infty} \left(\frac{q_j^2}{x_j} - \frac{q_j^2}{2T x_j} \right) f\left(x_j\right)\, dx_j. \tag{4}$$

and the expected shortage cost of the inventory system becomes,

$$q_j(t) = \int \left(- \frac{x_j}{T} \right) dt. \qquad (t_r < t < T)$$

With initial condition at $t = t_j,\ q_j(t_j) = 0$, The expected average inventory is given by

$$\int_{tr}^{T} \left(\frac{x_j}{T} \left(t - \frac{T q_j}{x_j} \right) \right) dt.$$

and the expected shortage cost is,

$$C_s \sum_{j=1}^{k} \int_{q_j}^{\infty} \left(\frac{T q_j^2}{2 x_j} - T q_j + \frac{T x_j}{2} \right) f\left(x_j\right)\, dx_j. \tag{5}$$

Now, the expected total cost per unit of time of the system consisting k^{th} buyers becomes,

$$TC_b = \sum_{j=1}^{k} \frac{O_j}{T} + \sum_{j=1}^{k} \frac{h_j}{T} \int_0^{q_j} \left(T q_j - \frac{T x_j}{2} \right) f\left(x_j\right) dx_j + \sum_{j=1}^{k} \frac{h_j}{T} \int_{q_j}^{\infty} \left(\frac{q_j^2}{x_j} - \frac{q_j^2}{2T x_j} \right) f\left(x_j\right)\, dx_j + \frac{C_s}{T}$$
$$\sum_{j=1}^{k} \int_{q_j}^{\infty} \left(\frac{T q_j^2}{2 x_j} - T q_j + \frac{T x_j}{2} \right) f\left(x_j\right)\, dx_j. \tag{6}$$

4.2 Manufacturer's costs

The manufacturer runs a single production setup to produce new and remanufactured products simultaneously and considers only one manufacturing-remanufacturing setup per production cycle to produce $Q = \sum_{j=1}^{k} q_j$ finished products. Therefore, the expected total costs of the manufacturer system (TC_m) comprises: ordering cost for ordering used products, setup cost, manufacturing and remanufacturing cost, quality up-gradation cost for finished products, holding costs, and goodwill lost cost.

4.2.1 Ordering cost (C_m^o)

The manufacturer orders to 3PL for collected products for which the fixed ordering cost per unit of time is $\frac{O_m}{T}$.

4.2.2 Setup cost (C_m^s)

The manufacturer produces finished products in one setup per production cycle for which the setup cost per cycle is S_m. This, the setup cost per unit time for manufacturer is obtained as; ($\frac{S_m}{T}$).

4.2.3 Manufacturer's inventory holding costs (C_m^h)

The manufacturer inventory consists of three components, finished product, used-products for remanufacturing, and, raw material inventory for new products which are calculated as given below.

Finished products inventory

The governing differential equations of manufacturer finished products instantaneous inventory $Q(t)$ is,

$$Q(t) = \int P_m dt. \qquad (0 < t < \ t_1)$$

Using initial conditions at ($t = 0,\ Q(0) = 0$), where t_1 is the production uptime, and manufacturer's finished product inventory after production time ($t = t_1$) gradually decreases as the manufacturer delivers only one shipment to a retailer at it's shipment time t_j, hence

$$\sum_{j=1}^{k} \int_{t_1}^{t_{j+1}} \left(-\frac{x_j}{T} \right) dt. \tag{7}$$

so, the inventory holding cost for finished products is

$$h_f \sum_{j=1}^{k} \int_0^{q_j} \left(\frac{q_j^2}{2P_m} - \int_{t_1}^{t_{j+1}} \frac{x_j}{T}\, dt \right) f\left(x_j\right) dx_j. \tag{8}$$

Used-product inventory

Manufacturer get EOL/EOU products from 3PL to fulfill a fraction of demand (θx_j) from remanufacturing. Until time θt_1, manufacturer consumes the used-products at a rate P_m, so the differential equation governing used-products inventory Q_u at time t is given as,

$$Q_u(t) = \int -P_m dt. \qquad (0 < t < \theta t_1) \tag{9}$$

With given condition $(t=0,\ Q_u=\theta T x_j)$, the expected inventory holding cost for used products is

$$h_u \sum_{j=1}^{k} \int_0^{q_j} \left(\frac{\theta^2 T q_j x_j}{P_m} - \frac{\theta^2 q_j^2}{2P_m} \right) f\left(x_j\right)\, dx_j. \tag{10}$$

Raw material inventory

As the manufacturer produces new items from virgin raw material, therefore the manufacturer has to maintain a fraction of demand, as $(1-\theta)x_j$ quantity of raw material. The raw material inventory depletes at a rate P_m, so governing differential equation governing eaw material inventory Q_r is,

$$Q_r(t) = \int -P_m dt. \qquad (\theta t_1 < t < t_1) \tag{11}$$

At time $(t=t_1,\ \ Q_r(t) = (q_j - P_m t)$, the expected holding cost for raw material inventory is,

$$h_r \sum_{j=1}^{k} \int_0^{q_j} \left(\frac{(\theta-1)^2 {q_j}^2}{2P_m} \right) f\left(x_j\right)\, dx_j. \tag{12}$$

4.2.4 Manufacturer's production cost (C_m^p)

The manufacturer consists of a hybrid manufacturing-manufacturing system and produced finished product from two types of input materials, (θx_j) quantity of used-product and $(1-\theta)x_j$ quantity of raw materials. Therefore, manufacturer bears two types of production costs. The manufacturing cost for new products which is calculated as,

$$C_m \sum_{j=1}^{k} \int_0^{q_j} (1-\theta) x_j f\left(x_j\right)\, dx_j. \tag{13}$$

The remanufacturing costs for used-products which is derived as,

$$C_{rm} \sum_{j=1}^{k} \int_0^{q_j} \left(\theta x_j\right) f\left(x_j\right)\, dx_j. \tag{14}$$

4.2.5 Manufacturer's quality improvement cost (C_m^q)

To get final product quality q_f from the average quality of used-products q_u, the manufacturer invest in quality upgradation during remanufacturing process.

$$C_q \sum_{j=1}^{k} \int_0^{q_j} \theta x_j \left(q_f^2 - q_u^2\right) f\left(x_j\right)\, dx_j. \tag{15}$$

The manufacturer also invest in manufacturing for quality up-gradation as,

$$C_q \sum_{j=1}^{k} \int_{0}^{q_j} x_j(1-\theta){q_f}^2 f\left(x_j\right) dx_j. \tag{16}$$

4.2.6 Manufacturer's goodwill loss cost (C_m^g)

The quality of finished products is not always 100%, because of mixed manufacturing and remanufacturing products, some contamination $(1-q_f)$ is adhered with final products. The quality imperfections affect the brand image of manufacturer and incur goodwill loss cost, as given below;

$$C_g \sum_{j=1}^{k} \int_{0}^{q_j} \left(1-q_f\right) x_j f(x_j) dx_j. \tag{17}$$

Hence, the total costs of manufacturer becomes;

$$\begin{aligned} TC_m = {} & \frac{O_m + S_m}{T} + \frac{h_f}{T} \sum_{j=1}^{k} \int_{0}^{q_j} \left(\frac{q_j^2}{2P_m} - \int_{t_1}^{t_{j+1}} \frac{x_j}{T}\, dt \right) f\left(x_j\right)\, dx_j + \frac{h_u}{T} \sum_{j=1}^{k} \int_{0}^{q_j} \left(\frac{\theta^2 T q_j x_j}{P_m} - \frac{\theta^2 q_j^2}{2P_m} \right) \\ & f\left(x_j\right) dx_j + \frac{h_r}{T} \sum_{j=1}^{k} \int_{0}^{q_j} \left(\frac{(\theta-1)^2 q_j^2}{2P_m} \right) f(x_j)\, dx_j + \frac{C_m}{T} \sum_{j=1}^{k} \int_{0}^{q_j} (1-\theta) x_j f\left(x_j\right)\, dx_j + \frac{C_{rm}}{T} \sum_{j=1}^{k} \int_{0}^{q_j} \\ & \left(\theta x_j\right) f x_j\, dx_j + \frac{C_q}{T} \sum_{j=1}^{k} \int_{0}^{q_j} \left(\theta x_j \left(q_f^2 - q_u^2\right)\right) f\left(x_j\right)\, dx_j + \frac{C_q}{T} \sum_{j=1}^{k} \int_{0}^{q_j} x_j (1-\theta) {q_f}^2 f\left(x_j\right)\, dx_j + \frac{C_g}{T} \\ & \sum_{j=1}^{k} \int_{0}^{q_j} \left(1-q_f\right) x_j f\left(x_j\right) dx_j. \end{aligned} \tag{18}$$

4.3 3PL costs TC_3

The total cost of 3PL consist of collection setup cost, investment for used product recovery, collected products inventory holding cost, collection cost, transportation cost, RTI management cost, RTI inventory holding cost, which are derived below;

4.3.1 3PL's setup cost (C_L^s)

The 3PL owns a collection center in a an off-site area to collect EOL/EOU items from market/customers, where recovered products undergo inspection and average acceptable quality $(\geq q_u)$ is acquired and deliver to manufacturer. The total setup cost per unit cycle for 3PL is obtained as $(\frac{S_c}{T})$.

4.3.2 3PL's inventory holding cost (C_L^h)

Collected products inventory

The 3PL collection center collects and accumulates EOL/EOU items to fulfill manufacturing center requests for delivery. Amid the cycle length T, the used production inventory builds up at a rate (θx_j), and depletes to zero at the beginning of production cycle. The collected product inventory is calculated as;

$$q_j(t) = \int \left(\frac{\theta x_j}{T} \right) dt. \qquad (0 < t < T) \tag{19}$$

Using initial condition at time $t = 0 \;\; q_j(0) = 0$, the total expected collected-inventory holding cost becomes

$$h_c \sum_{j=1}^{k} \int_0^{q_j} \left(\frac{\theta x_j T}{2} \right) f\left(x_j\right) \, dx_j. \tag{20}$$

4.3.3 3PL's collection cost (C_L^c)

At collection center, 3PL incurs an average recycling cost C_{rc}, to collect used products from customer and inspect to get an accepted quality level q_u. As, the total collected amount depends on a certain fraction of demand (θx_j), so the total collection cost is derived as follow;

$$C_{rc} \sum_{j=1}^{k} \int_0^{q_j} (\theta x_j) \, f\left(x_j\right) \, dx_j. \tag{21}$$

4.3.4 3PL's transportation cost (C_L^t)

As the 3PL owns a collection center and indulge in closed-loop logistic activities, we consider the transport cost for 3PL based on;

(1) transport vehicle capacity (RTI container based on number of products),

(2) transport distances between all players of supply chain,

(3) transportation cost per RTI container per unit distance C_t.

The transportation cost of closed-loop logistics has three components, the transportation cost between manufacturer and buyers $(C_t \sum_{j=1}^{k} x_j l_{mj})$ for transporting finished products and, the transportation cost for collecting used product from diversely located customers by collection center, $\sum_{j=1}^{k} \theta x_j l_{cj}$, and finally transporting recovered products from collection-center to manufacturing unit $\sum_{j=1}^{k} \theta x_j l_{mc}$.

$$C_t \sum_{j=1}^{k} \int_0^{q_j} \left(\frac{\theta l_{cj} x_j}{\tau} + \frac{\theta l_{cm} x_j}{\tau} + \frac{x_j l_{jm}}{\tau} \right) f\left(x_j\right) \, dx_j. \tag{22}$$

4.3.5 3PL's container management cost (C_L^m)

The required number of RTIs is an important factor, as the cost of system increases with an increased number of RTI packaging. Moreover, less than required number of containers may cause material flow very disruptive and unproductive. Therefore, we consider the minimum required number of containers as equal to fulfil the largest shipment at one time. Hence, the container management cost becomes;

$$C_\tau \sum_{j=1}^{k} \int_0^{q_j} \left(x_j \tau^{s-1}\right) f\left(x_j\right) \, dx_j. \tag{23}$$

where, s is the scaling parameter which represents the correlation between a container capacity and its required management cost.

4.3.6 3PL's container holding cost (C_L^{hc})

As the 3PL holds the RTIs at it's setup to make deliveries in the CLSC, the inventory holding cost for containers is;

$$h_R \sum_{j=1}^{k} \int_0^{q_j} \left(\frac{x_j \left(T - l_j\right)}{\tau T}\right) f\left(x_j\right) \, dx_j. \tag{24}$$

Hence, the total cost of 3PL become;

$$\begin{aligned} TC_3 \quad &= \frac{S_c}{T} + \frac{\gamma \theta^2}{T} + \frac{h_c}{T} \sum_{j=1}^{k} \int_0^{q_j} \frac{1}{2} \left(\theta x_j T\right) f\left(x_j\right) \, dx_j + \frac{C_{rc}}{T} \sum_{j=1}^{k} \int_0^{q_j} (\theta x_j) f\left(x_j\right) \, dx_j + \frac{C_t}{T} \sum_{j=1}^{k} \\ & \int_0^{q_j} \left(\frac{\theta x_j l_{cj}}{\tau} + \frac{\theta l_{cm} x_j}{\tau} + \frac{x_j l_{jm}}{\tau}\right) f\left(x_j\right) \, dx_j + \frac{C_\tau}{T} \sum_{j=1}^{k} \int_0^{q_j} \left(x_j \tau^{s-1}\right) f\left(x_j\right) \, dx_j + \frac{h_R}{T} \\ & \sum_{j=1}^{k} \int_0^{q_j} \left(\frac{x_j \left(T - l_j\right)}{\tau T}\right) f\left(x_j\right) \, dx_j. \end{aligned} \tag{25}$$

4.4 Carbon emissions costs TC_e

The study also consider carbon emissions costs associated with closed-loop logistic system. The total carbon emissions cost of closed-loop logistic systems consist on emissions costs due to transportation activities based on transport distances follows;

(1) transporting finished products from manufacturer to buyers $\frac{C_{ec} e_r g_m (\theta x_j l_{jm})}{t_c}$,

(2) collecting product from diversely located customers at collection center $\frac{C_{ec} e_r g_m (\theta x_j l_{cj})}{t_c}$,

(3) transporting collected EOL/EOU products to manufacturing unit as $\frac{C_{ec} e_r g_m (\theta l_{cm} x_j)}{t_c}$.

(4) The study also consider emissions due to production activities at the manufacturing unit ($C_{ec} e_m x_j$), where

emissions cost based on production activities are referred as (emissions per product × the number of products produced) and, total emissions costs are considered given below,

$$\sum_{j=1}^{k}\int_0^{q_j}\left(\frac{C_{ec}e_r g_m\left(\theta l_{cm}x_j\right)}{t_c}+\frac{C_{ec}e_r g_m\left(\theta x_j l_{cj}\right)}{t_c}+\frac{C_{ec}e_r g_m\left(\theta x_j l_{jm}\right)}{t_c}+C_{ec}e_m x_j\right)f\left(x_j\right)\,dx_j. \tag{26}$$

4.5 Expected cost of the proposed supply chain

Total expected cost of supply chain ($TCSC$) is the sum of buyer's cost, manufacturer's cost, $3PL$'s costs and emissions cost of the system, which is given as,

$$TCSC\left(q_j,\theta,\tau\right)=TC_b+TC_m+TC_L+TC_e. \tag{27}$$

More comprehensively, Eq. (28) provides the expected total cost of CLSC.

$$\begin{aligned}
TCSC\left(q_j,\theta,\tau\right)&=\sum_{j=1}^{k}\frac{O_j}{T}+\sum_{j=1}^{k}\frac{h_j}{T}\int_0^{q_j}\left(Tq_j-\frac{Tx_j}{2}\right)f\left(x_j\right)\,dx_j+\sum_{j=1}^{k}\frac{h_j}{T}\int_{q_j}^{\infty}\left(\frac{q_j^2}{x_j}-\frac{q_j^2}{2Tx_j}\right)\\
&f\left(x_j\right)\,dx_j+\frac{C_s}{T}\sum_{j=1}^{k}\int_{q_j}^{\infty}\left(\frac{Tq_j^2}{2x_j}-Tq_j+\frac{Tx_j}{2}\right)f\left(x_j\right)dx_j+\frac{C_g}{T}\sum_{j=1}^{k}\int_0^{q_j}((1-q_f)\,x_j)f\left(x_j\right)\,dx_j\\
&+\frac{S_c}{T}+\frac{\gamma\theta^2}{T}+\frac{h_c}{T}\sum_{j=1}^{k}\int_0^{q_j}\frac{1}{2}\left(\theta x_j T\right)f\left(x_j\right)\,dx_j+\frac{C_{rc}}{T}\sum_{j=1}^{k}\int_0^{q_j}\left(\theta x_j\right)f\left(x_j\right)\,dx_j+\frac{C_t}{T}\sum_{j=1}^{k}\int_0^{q_j}\\
&\left(\frac{\theta x_j l_{cj}}{\tau}+\frac{\theta l_{cm}x_j}{\tau}+\frac{x_j l_{jm}}{\tau}\right)f\left(x_j\right)\,dx_j+\frac{C_\tau}{T}\sum_{j=1}^{k}\int_0^{q_j}\left(x_j\tau^{s-1}\right)f\left(x_j\right)\,dx_j+\frac{h_R}{T}\sum_{j=1}^{k}\int_0^{q_j}\Bigg(\frac{x_j}{\tau T}\\
&\left(T-l_j\right)\Bigg)f\left(x_j\right)\,dx_j+\frac{O_m+S_m}{T}+\frac{h_f}{T}\sum_{j=1}^{k}\int_0^{q_j}\left(\frac{q_j^2}{2P_m}-\int_{t_1}^{t_{j+1}}\frac{x_j}{T}\,dt\right)f\left(x_j\right)\,dx_j+\frac{h_u}{T}\sum_{j=1}^{k}\\
&\int_0^{q_j}\left(\frac{\theta^2Tq_jx_j}{P_m}-\frac{\theta^2q_j^2}{2P_m}\right)f\left(x_j\right)\,dx_j+\frac{h_r}{T}\sum_{j=1}^{k}\int_0^{q_j}\left(\frac{(\theta-1)^2q_j^2}{2P_m}\right)f(x_j)\,dx_j+\frac{C_m}{T}\sum_{j=1}^{k}\int_0^{q_j}((1-\theta)\\
&x_j)f\left(x_j\right)dx_j+\frac{C_{rm}}{T}\sum_{j=1}^{k}\int_0^{q_j}\left(\theta x_j\right)fx_j\,dx_j+\frac{C_q}{T}\sum_{j=1}^{k}\int_0^{q_j}\left(\theta x_j\left(q_f^2-q_u^2\right)\right)f\left(x_j\right)\,dx_j+\frac{C_q}{T}\sum_{j=1}^{k}\\
&\int_0^{q_j}(1-\theta)q_f^2(x_j)f\left(x_j\right)\,dx_j.
\end{aligned} \tag{28}$$

4.6 Uniform distribution

Until now, the model considers stochastic demand with general probability distribution. Now, the model is analyzed with a specific distribution function that the demand follows. Hence, if demand pursues a uniform distribution with parameter a_j and b_j, then the probability density function $f(x_j)$ is provided as,

$$f(r) = \begin{cases} \frac{1}{b_j - a_j} & a_j \leq x_j \leq b_j \\ \\ 0 & otherwise \end{cases}$$

Now, the expected total cost $TCSC\left(q_j, \theta, \tau\right)$ of CLSC becomes,

$$\begin{aligned} TCSC(q_j,\tau,\theta) = \frac{1}{T}\Bigg(& \sum_{j=1}^{k}(O_j + S_m + S_r + O_m) - \sum_{j=1}^{k} \frac{h_j\left(a_j - 3q_j\right)\left(a_j - q_j\right)}{4\left(a_j - b_j\right)} + \\ & h_c \sum_{j=1}^{k} \frac{\theta T\left(a_j^2 - q_j^2\right)}{4\left(a_j - b_j\right)} + h_f \sum_{j=1}^{k} \frac{\left(a_1 - q_1\right)\left(a_1 P_m\left(t_1 - t_2\right) + P_m q_1\left(t_1 - t_2\right) + q_1^2 T\right)}{2 P_m T\left(a_1 - b_1\right)} + \\ & h_u \sum_{j=1}^{k} \frac{\theta^2 q_j\left(a_j - q_j\right)\left(T a_j + (T-1) q_j\right)}{2 P_m\left(a_j - b_j\right)} + h_r \sum_{j=1}^{k} \frac{1}{2P\left(a_j - b_j\right)}\left((\theta - 1)^2 q_j\left(a_j - q_j\right)\right. \\ & \left.\left(T a_j + (T-1) q_j\right)\right) - \sum_{j=1}^{k} \frac{q_j\left(a_j - q_j\right)\left((\theta - 1) C_m - \theta\left(C_r + C_{rm}\right)\right)}{a_j - b_j} + \sum_{j=1}^{k} C_t\left(a_j - q_j\right) \\ & \frac{\left(q_j\left(\theta l_{cj} + l_{jm}\right) + \theta l_{cm} q_j\right)}{a_j - b_j} + \sum_{j=1}^{k} \frac{C_q q_j\left(a_j - q_j\right)\left(q_f^2 - \theta q_u^2\right)}{a_j - b_j} + \gamma\theta^2 + \sum_{j=1}^{k} C_{ec} q_j\left(\theta e_r g_m\right. \\ & \left.\frac{\left(a_j - q_j\right)\left(l_{cj} + l_{cm} + l_{jm}\right) + t_c e_m\right)}{t_c\left(a_j - b_j\right)}\Bigg). \end{aligned} \tag{29}$$

4.7 Solution methodology

The optimal solutions of cost function (28) is derived as follow:

Assuming q_j ($\forall\ j = 1, 2, 3..., k$), τ, and θ are any real non-zero numbers, then, there $\exists$ a unique ${q_j}^*$ ($\forall\ j = 1, 2, 3, ..., k$), τ^* and θ^* that minimizes (28) and fulfils the given first-order conditions (FOC):

$$\frac{\partial\left(TCSC\left(q_j, \tau, \theta\right)\right)}{\partial q_j} = 0, \tag{30}$$

$$\frac{\partial\left(TCSC\left(q_j, \tau, \theta\right)\right)}{\partial \theta} = 0, \tag{31}$$

$$\frac{\partial\left(TCSC\left(q_j, \tau, \theta\right)\right)}{\partial \tau} = 0. \tag{32}$$

Furthermore, $TCSC\left(q_j, \tau, \theta\right)$ is convex, if and only if at ${q_j}^*$ ($\forall\ j = 1, 2...k$), τ^* and θ^* the Hessian of $TCSC\left(q_j, \tau, \theta\right)$ is positive semi-definite, proof given in Appendix A.2.

5 Numerical experiments

The mathematical model developed for the proposed system is analysed with different numerical experiments and sensitivity analysis. The values of input parameters are provided in Appendix A.1. The numerical examples consider a three-echelon supply chain model that comprises buyers, a manufacturer, and a 3PL. Four different cases are considered, in which Case 1 studies a single retailer, single manufacturer, and a single 3PL scenario. Whereas the other three cases analyse multi-retailers scenarios. Case 2 studies two retailer, Case 3 studies three retailers, and finally Case 4 studies four retailers with a manufacturer and 3PL. For each of the case, parameters related to the previous case are kept constant whereas new parameters have been added to develop the multi-retailer cases.

Table 2: Optimal results of the 4 Cases

Case 1	Retailer optimal decisions				
	Retailer number (j)	1			
	lot size q_j (units)	8.477			
	Manufacturer and 3PL optimal decisions				
	lot size Q= $\sum q_j$(units)	8.477			
	Remanufacturing rate θ	0.095			
	Container capacity τ	6 units			
Case 2	Retailer optimal decisions				
	Retailer number (j)	1	2		
	lot size q_j (units)	7.752	6.104		
	Manufacturer and 3PL optimal decisions				
	lot size Q= $\sum q_j$(units)	13.857			
	Remanufacturing rate θ	0.220			
	Container capacity τ	6 units			
Case 3	Retailer optimal decisions				
	Retailer number (j)	1	2	3	
	lot size q_j (units)	5.779	5.113	10.751	
	Manufacturer and 3PL optimal decisions				
	lot size Q= $\sum q_j$(units)	21.644			
	Remanufacturing rate θ	0.264			
	Container capacity τ	5 units			
Case 4	Retailer optimal decisions				
	Retailer number (j)	1	2	3	4
	lot size q_j (units)	5.76	5.07	10.68	4.28
	Manufacturer and 3PL optimal decisions				
	lot size Q= $\sum q_j$(units)	25.81			
	Remanufacturing rate θ	0.33			
	Container capacity τ	6 units			

Case 2 comprises two retailers, Case 3 comprises three retailers, and Case 4 studies a supply chain with four different retailers. The optimal results are provided in Table 2, which show that an optimal policy is the hybrid strategy with both remanufacturing and manufacturing for all four cases; however, the rate of remanufacturing increases from Case 1 to Case 4. The increasing value is because of the increasing value in manufacturer's lot size $Q = \sum q_j$. For Case 1, it is too low because the fixed management costs associated with remanufacturing are too high and, therefore, the model shifts towards a higher manufacturing policy. As the manufacturer's lot size increases, the model opts for a higher remanufacturing rate. To dig deep into this relationship, a special case is considered for Case 1. In the special case, the demand of retailer 1 is increased 10 folds, such that the parameter $a_1 = 40$ and $b_1 = 100$. The optimal value for remanufacturing now increases to 15% from the previous 0.95%. To study the relationship between remanufacturing rate and manufacturer lot size, the special case for pure-manufacturing policy is also considered. The results show that the lot size decreases for pure-manufacturing policy. This result leads to an intuition that the remanufacturing increases the cycle time of the supply chain and, thus, it decreases the setup and ordering cost per cycle.

5.1 Sensitivity analysis

A sensitivity analysis is carried out for all the input parameters of the model such that one parameter is changed at a time and the all other parameters are kept constant. The value of each parameter is changed by -50%, -25%, $+25\%$, and $+50\%$. The results of sensitivity analysis are arranged in Tables 3, 4 and 5. Table 3 provides a sensitivity analysis of the factors associated with manufacture and 3PL. Because Case 4 is the most complex case among all the developed cases and other cases are sub cases of Case 4, therefore, for further sensitivity analysis we consider Case 4.

From the numerical outcomes, the following important understandings are obtained:

- The dominant parameter is manufacturing cost. Reduction in the manufacturing cost decreases the expected total cost of CLSC. However, the percentage change in the expected cost to increment and reduction in cost of manufacturing is highly asymmetric. Reduction in the manufacturing cost by 50 percent decreases the expected profit by 22.42%; on the contrary, an equal increase raises the cost by only 6.24%. Therefore, it can be stated that the system cost is highly sensitive to reduction in manufacturing cost as compared to the increments. Thus, supply chain managers should ponder upon technology advancements to reduce cost of manufacturing (new products) in order to improve overall profitability in hybrid systems. Increasing manufacturing cost shifts the system to remanufacturing process and for 50% increase the optimal remanufacturing reaches to 100%.

- Moreover, comparable results to manufactuirng cost are shown by remanufacturing cost. Lowering remanufacturing cost by 50% reduces the expected cost by 9.28%, whereas an increment in remanufacturing cost by 50% reduces profit by 1.67%. Decreasing remanufacturing cost increase the optimal remanufacturing percentage, and for 50% decrease, the remanufactruing percentage is 100%. However, increasing the remanufacturing cost does not reduce remanufacturing percentage with a similar proportion, although for 50% increase the model shifts towards a manufacturing only policy. Furthermore, with the increase in the remanufacturing cost, the optimal production quantities are also reduced. This means, remanufacturing is agitated by the economy of scale.

Table 3: Sensitivity analysis of the manufacturer's and 3PL's parameters

Parameter	% change	q_1	q_2	q_3	q_4	τ	θ	% change in $TCSC$
h_r	-50	5.77	5.07	10.69	4.28	6	0.33	$+0.02$
	-25	5.77	5.07	10.69	4.28	6	0.33	$+0.01$
	$+25$	5.76	5.07	10.68	4.28	6	0.33	-0.01
	$+50$	5.75	5.07	10.67	4.28	6	0.33	-0.02
h_R	-50	5.77	5.07	10.70	4.28	6	0.33	-0.08
	-25	5.77	5.07	10.69	4.28	6	0.33	-0.04
	$+25$	5.76	5.07	10.68	4.28	6	0.33	$+0.04$
	$+50$	5.76	5.06	10.67	4.28	6	0.33	$+0.08$
Cq	-50	5.80	5.11	10.77	4.32	6	0.33	-0.70
	-25	5.78	5.09	10.73	4.30	6	0.33	-0.35
	$+25$	5.74	5.05	10.64	4.26	6	0.33	$+0.35$
	$+50$	5.72	5.02	10.60	4.25	6	0.32	$+0.69$
g_m	-50	5.78	5.08	10.71	4.29	6	0.34	-0.08
	-25	5.77	5.08	10.70	4.29	6	0.33	-0.04
	$+25$	5.75	5.06	10.67	4.27	6	0.32	$+0.04$
	$+50$	5.74	5.05	10.66	4.27	6	0.32	$+0.08$
e_r	-50	5.78	5.08	10.71	4.29	6	0.34	-0.08
	-25	5.77	5.08	10.70	4.29	6	0.33	-0.04
	$+25$	5.75	5.06	10.67	4.27	6	0.32	$+0.04$
	$+50$	5.74	5.05	10.66	4.27	6	0.32	$+0.08$
t_c	-50	5.73	5.04	10.63	4.25	6	0.31	$+0.16$
	-25	5.75	5.06	10.66	4.27	6	0.32	$+0.05$
	$+25$	5.77	5.07	10.70	4.29	6	0.33	-0.03
	$+50$	5.77	5.08	10.70	4.29	6	0.33	-0.05
h_u	-50	5.76	5.07	10.68	4.28	6	0.33	0.00
	-25	5.76	5.07	10.68	4.28	6	0.33	0.00
	$+25$	5.76	5.07	10.68	4.28	6	0.33	0.00
	$+50$	5.76	5.07	10.69	4.28	6	0.33	0.00
Cg	-50	5.76	5.07	10.68	4.28	6	0.33	0.00
	-25	5.76	5.07	10.68	4.28	6	0.33	0.00
	$+25$	5.76	5.07	10.68	4.28	6	0.33	0.00
	$+50$	5.76	5.07	10.68	4.28	6	0.33	0.00
C_{ec}	-50	5.78	5.09	10.72	4.30	6	0.34	-0.11
	-25	5.77	5.08	10.70	4.29	6	0.33	-0.05
	$+25$	5.77	5.06	10.67	4.27	6	0.32	$+0.05$
	$+50$	5.74	5.05	10.65	4.27	6	0.32	$+0.10$
h_f	-50	5.78	5.07	10.71	4.28	6	0.33	$+0.15$
	-25	5.77	5.07	10.70	4.28	6	0.33	$+0.07$
	$+25$	5.75	5.07	10.67	4.28	6	0.33	-0.07
	$+50$	5.74	5.07	10.66	4.28	6	0.33	-0.15
h_c	-50	5.75	5.06	10.66	4.27	6	0.31	$+0.14$
	-25	5.76	5.06	10.67	4.28	6	0.32	$+0.07$
	$+25$	5.77	5.07	10.69	4.28	6	0.33	-0.07
	$+50$	5.77	5.08	10.70	4.29	6	0.34	-0.15

- For manufacturer's setup cost, the model showed almost symmetrical results. The expected total cost decreases by 4.66% for a 50% decreases in S_m, where increasing setup cost by 50% raises the expected cost by 4.40%. The ordering cost of the manufacturer O_m also has the same effect on the expected cost. 3PL setup cost S_c also has symmetric effects with around 3.7% decrease and 3.54% increase in the expected cost.
- Container management cost produces asymmetric changes in the expected cost; a 50% increase raises the expected cost by 2.55%, and a 50% decrease reduces the expected cost by 1.87%.
- Similarly, the investment cost to collect used product also produces asymmetric changes; for 25% change in the investment cost the expected cost is effected symmetrically but for 50% change in the cost the expected cost becomes more sensitive towards negative changes.

- Similarly, the investment cost to collect used product also produces asymmetric changes; for 25% change in the investment cost the expected cost is effected symmetrically but for 50% change in the cost the expected cost becomes more sensitive towards negative changes.

- The transportation cost and recycling cost also produces asymmetric changes and the expected cost shows a high sensitivity towards negative changes in both the parameters.

Sensitivity analysis of all other factors related to manufacturer and 3PL is presented in Table 4. Table 5 shows the result of sensitivity analysis for the retailer's parameters. Considering the retailer's cost parameters, the most important one is the ordering cost O_j of retailers. Decreasing it by 50% reduces the total cost by 11.01% and increasing it by 50% raises the cost by 9.65%. both the ordering quantity and remanufacturing percentage increases with an increase in retailer ordering cost. The demand parameter a_j and b_j has a very high impact on expected cost, this is because with decreasing a_j the expected value of the demand also decreases and, therefore, the expected cost is decreased. However, increase of a_j produces infeasible results because, in this case, a_j becomes greater than b_j; similar results are obtained for b_j also.

5.1.1 Comparative study

The impacts of non-optimal policies on the expected total cost of the system are discussed in this section. The impact of retailer's lot size on expected cost is shown in Figure 3.

Figure 3 shows that as retailer's lot size increases, the $TCSC(q_j, \theta, \tau)$ of the system decreases to the optimal quantity where the total cost is minimum. Moreover, above the optimal value the expected cost of the system increases for retailer's lot size. The graph shows that the expected is not monotonically decreasing in q_j, and there exists an optimal production lot size q_j, after which the total expected costs start to increase with an increase in production quantity. The expected total cost shows convexity in RTI capacity where the impact of RTI capacity

Table 4: Sensitivity analysis of the manufacturer's and 3PL's parameters

Parameter	% change	q_1	q_2	q_3	q_4	τ	θ	% change in $TCSC$
C_m	-50	6.82	6.44	12.99	5.34	5	0.00	-22.42
	-25	6.04	5.48	11.33	4.61	5	0.00	-9.56
	$+25$	6.93	5.90	12.68	4.97	7	0.90	$+5.36$
	$+50$	7.11	6.11	13.21	5.07	5	0.99	$+6.24$
C_r	-50	8.11	7.13	15.03	5.90	7	1.00	-9.28
	-25	6.43	5.67	11.96	4.75	6	0.60	-2.66
	$+25$	5.50	4.85	10.22	4.09	5	0.14	$+1.27$
	$+50$	5.45	4.79	10.09	4.06	5	0.00	$+1.67$
S_m	-50	5.35	4.83	10.04	4.06	6	0.29	-4.66
	-25	5.56	4.95	10.37	4.17	6	0.31	-2.29
	$+25$	5.96	5.18	10.99	4.38	6	0.35	$+2.23$
	$+50$	6.15	5.29	11.28	4.49	6	0.36	$+4.40$
O_m	-50	6.15	5.29	11.28	4.49	6	0.36	-4.40
	-25	5.72	5.05	10.62	4.26	6	0.32	-0.45
	$+25$	5.80	5.09	10.75	4.30	6	0.33	$+0.45$
	$+50$	5.84	5.11	10.81	4.32	6	0.33	$+0.90$
S_c	-50	5.44	4.88	10.18	4.10	6	0.30	-3.71
	-25	5.60	4.97	10.43	4.19	6	0.31	-1.83
	$+25$	5.92	5.16	10.93	4.36	6	0.34	$+1.79$
	$+50$	6.07	5.25	11.17	4.45	6	0.36	$+3.54$
C_τ	-50	6.00	5.27	11.07	4.47	8	0.38	-2.55
	-25	5.87	5.16	10.85	4.37	7	0.35	-1.14
	$+25$	5.66	4.99	10.54	4.20	5	0.30	$+0.96$
	$+50$	5.61	4.93	10.43	4.15	5	0.29	$+1.87$
γ	-50	6.36	5.60	11.84	4.70	7	0.84	-2.31
	-25	5.89	5.20	10.96	4.37	6	0.46	-0.69
	$+25$	5.65	4.99	10.52	4.20	5	0.24	$+0.38$
	$+50$	5.62	4.95	10.44	4.18	5	0.20	$+0.61$
Ct	-50	5.99	5.26	11.05	4.46	4	0.38	-2.41
	-25	5.86	5.15	10.84	4.36	5	0.35	-1.10
	$+25$	5.65	4.99	10.53	4.19	6	0.30	$+0.92$
	$+50$	5.59	4.93	10.43	4.15	7	0.29	$+1.76$
C_{rc}	-50	6.11	5.39	11.36	4.53	6	0.48	-1.55
	-25	5.91	5.20	10.96	4.38	6	0.40	-0.69
	$+25$	5.66	4.97	10.48	4.20	6	0.27	$+0.55$
	$+50$	5.55	4.90	10.32	4.13	5	0.19	$+0.96$

Table 5: Sensitivity analysis for parameters related to retailers

Parameter	% change	q_1	q_2	q_3	q_4	τ	θ	% change in $TCSC$
h_j	-50	6.43	5.27	11.5	4.51	6	0.3	$+00.14$
	-25	6.10	5.17	11.1	4.40	6	0.3	$+00.13$
	$+25$	5.42	4.96	10.2	4.16	6	0.3	-00.26
	$+50$	5.06	4.85	9.77	4.03	6	0.2	-00.67
O_j	-50	4.75	4.48	9.14	3.73	5	0.23	-11.01
	-25	5.29	4.79	9.96	4.03	6	0.29	-05.29
	$+25$	6.20	5.32	11.36	4.51	6	0.37	$+04.96$
	$+50$	6.61	5.5	11.99	4.73	6	0.41	$+09.65$
Cs	-50	5.25	4.21	9.46	3.60	5	0.24	-06.31
	-25	5.52	4.65	10.07	3.95	6	0.29	-02.94
	$+25$	6.01	5.47	11.29	4.60	6	0.37	$+02.55$
	$+50$	6.25	5.86	11.87	4.89	6	0.41	$+04.77$
a_j	-50	5.74	7.47	12.86	5.97	5	0.21	-43.76
	-25	5.76	6.46	11.95	5.27	6	0.26	-28.35
	$+25$	inf	inf	inf	inf	inf	inf	inf
	$+50$	inf	inf	inf	inf	inf	inf	inf
b_j	-50	inf	inf	inf	inf	inf	inf	inf
	-25	inf	inf	inf	inf	inf	inf	inf
	$+25$	6.50	7.20	13.42	5.87	6	0.29	-13.96
	$+50$	7.22	8.89	15.75	7.13	5	0.26	-17.29

inf=infeasible results

τ on expected cost is shown in Figure 4(a). With the increase in RTI capacity, the expected cost decreases to a optimal limit, after which it tends to increase again.

Figure 4(b) presents the effects of remanufacturing rate on the expected cost of the system. It is clear that $TCSC(q_j, \theta, \tau)$ is convex in θ, even though, the cost of remanufacturing is half the cost of manufacturing, the optimal policy is the hybrid one. In the existing literature many studies, for example Sarkar et al. (2017b), Naeem et al. (2013), and Dobos and Richter (2004), pointed out this phenomenon. Other models, for example, Dobos and Richter (2006) and Richter (1996) stated that pure policies are more cost-effective compared to the mixed policy.

Figure 5 shows the impact of lot size of 4 retailers and container capacity on $TCSC(q_j, \theta, \tau)$, it is clear from the figure that the expected cost initially decreases in both the lot size and container capacity and then increases with respect to both parameters for all retailers.

Figure 6 shows the impact of lot size of 4 retailers and remanufacturing rate on $TCSC(q_j, \theta, \tau)$. The expected total cost is convex in both the lot size and remanufacturing rate for all retailers. Figure 7 shows the impact of remanufacturing rate and container capacity on $TCSC(q_j, \theta, \tau)$. The expected total cost increases in both directions of the container capacity and remanufacturing rate.

5.2 managerial Insights

The following insights for managers have been drawn from the results:

- The full benefits of remanufacturing are associated with the economy of scale. Remanufacturing is highly

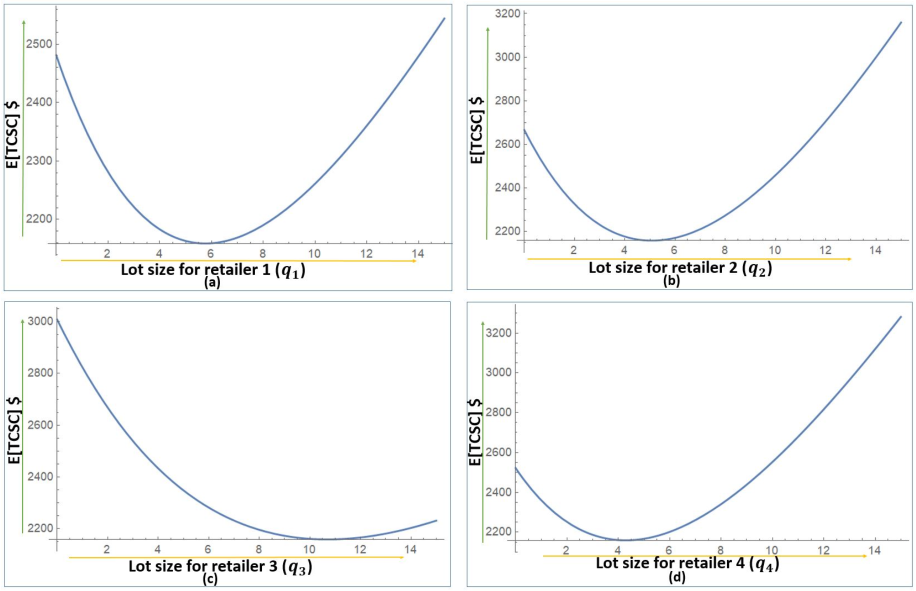

Figure 3: Impact of retailer's lot size on $TCSC(q_j, \theta, \tau)$ of the system: (a) retailer 1, (b) retailer 2, (c) retailer 3, (d) retailer 4,

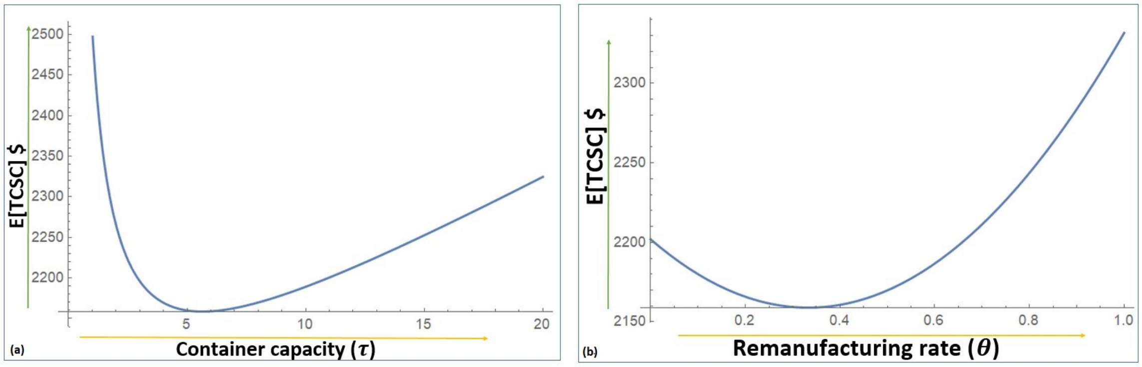

Figure 4: Impact of container capacity and remanufacturing rate on $TCSC(q_j, \theta, \tau)$ of the system: (a) container capacity, (b) remanufacturing rate

profitable for systems with high demand irrespective of slight changes in remanufacturing costs. The environmental benefits are a plus in this case. On the other hand, low demand/recovery-rates cannot balance the costs and environmental deterioration of RL activities. Hence, for products with low demand/recovery-rate, remanufacturing is not an optimal strategy.

- We obtain another important insight for systems with high fixed costs per order/setup, such as setup cost, ordering cost, fixed transportation cost. etc.. The results show that the lot size increases with an increase in the remanufacturing rate, and, for constant demand, the cycle time is proportional to lot size. Thus the number of setups/orders per unit time is decreased with an increase in lot size. Therefore, managers facing high setup/ordering costs can also use remanufacturing is a tool to reduce the total cost of the system.

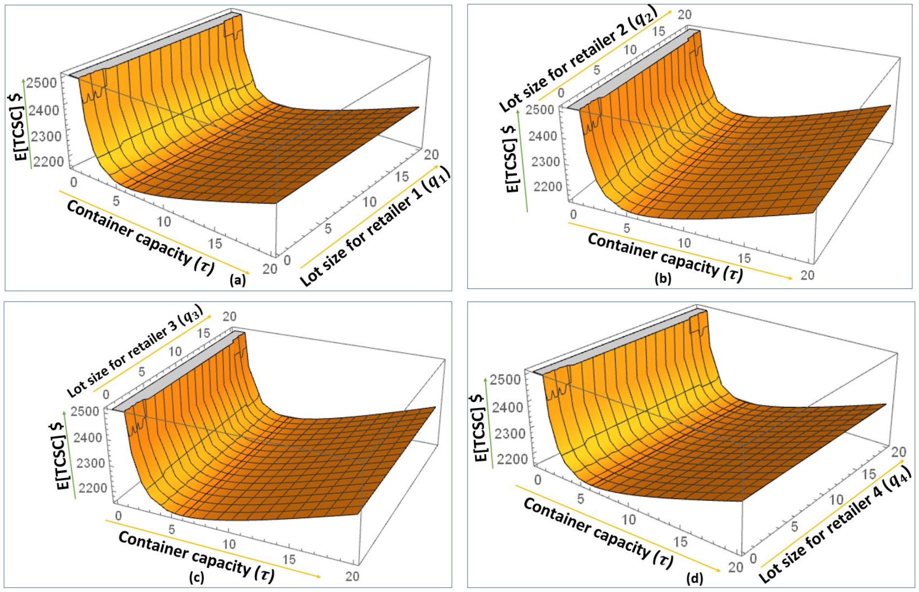

Figure 5: Joint impact of container capacity and retailer's lot size on $TCSC(q_j, \theta, \tau)$ of the system: (a) retailer 1, (b) retailer 2, (c) retailer 3, (d) retailer 4,

- The two most important parameters that decide the optimal percentage of remanufacturing are the manufacturing cost and remanufacturing cost. A wrong calculation of one of them would lead to suboptimal or non-optimal results. However, interlinking them with each other can reduce the effects of errors in calculation. Here, the errors may only lead to suboptimal results instead of non-optimal ones. Hence, we suggest managers use remanufacturing costs as a percentage of manufacturing cost instead of using a unique value for remanufacturing costs.

- The results further show that the remanufacturing rate is decreasing in both transportation costs and distances. Hence, the profitability of remanufacturing and RL depends upon the cost of transporting used products and the distance of used product transportation. Therefore, the managers should avoid remanufacturing of products if transportation distances are large. Sarkar et al. (2017b) noted similar observations regarding the relationship between transportation and remanufacturing.

6 Conclusions

In today's production industry, stochastic supply chain management model is more practical than the deterministic supply chain model as a consequence of the uncertainties in market demand. It is accessible and more realistic to forecast the probability distribution of demands instead of determining the definite values of market demands. The aim of this study was to develop the optimal inventory and RTI design policy for a multi-echelon CLSC model under stochastic demand. In order to minimize the environmental effects of the supply chain, the proposed model considered remanufacturing, RTI, and carbon discharge/emissions from production and transportation activities.

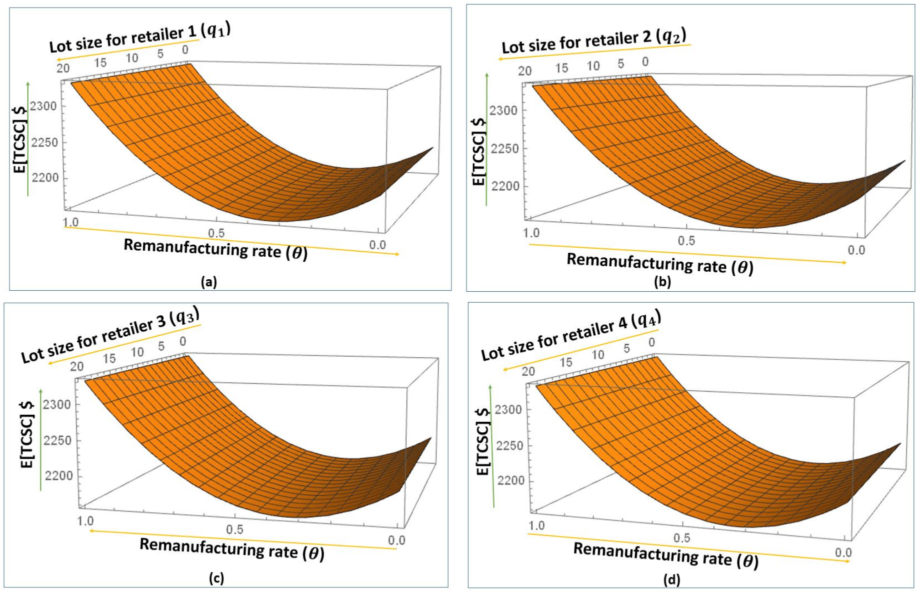

Figure 6: Joint impact of remanufacturing rate and retailer's lot size on $TCSC(q_j, \theta, \tau)$ of the system: (a) retailer 1, (b) retailer 2, (c) retailer 3, (d) retailer 4,

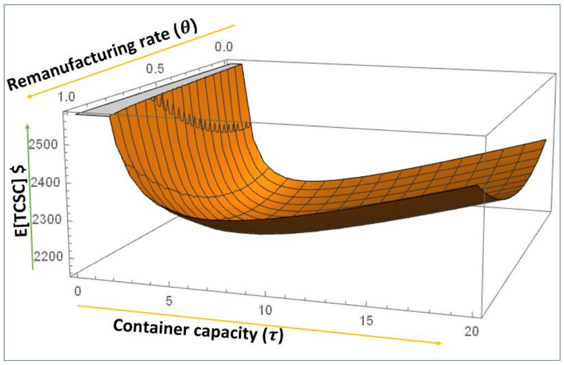

Figure 7: Joint impact of remanufacturing rate and container capacity on $TCSC(q_j, \theta, \tau)$ of the system

The remanufacturing reduced the use of natural resources and RTI decreased supply chain solid waste generation from the packaging. Moreover, to improve the sustainable performance of supply chain management, carbon discharge from production-logistic activities are also considered. The optimal order quantity for retailers, RTI container capacity, and the remanufacturing rate were obtained.

The results indicated that a hybrid strategy was the optimal one. The results further provided that manufacturing cost, remanufacturing cost and retailer's ordering cost were the most important parameters in CLSC models with stochastic parameters. The reduction in these parameters produced higher positive changes in comparison to the negative changes caused by their increments. An unusual relationship between remanufacturing and setup or ordering cost of the system was obtained. The optimal order quantities were increasing in the remanufacturing rate. Two insights could be drawn from this observation: 1) remanufacturing was inspired by economies of scale, and 2) remanufacturing could reduce setup and ordering cost of the supply chain. While the first one was more general, the second was interesting for supply chain managers having higher ordering and setup costs. They could use remanufacturing to increase their cycle time or ordering quantities and thus, reduced the number of orders per year. For such supply chains, remanufacturing could even be proven profitable with a higher remanufacturing cost, because the supply chain would reduce considerable cost from reducing the number of order or setup per year. Moreover, product take-back and remanufacturing activities develop supply uncertainty into the supply chain owing to uncertain returns which in turn leads to uncertain production quantities. Thus, a prompt extension of this study is to consider stochastic returns and production rate. However, in that case, the complexity of the model would increase many times and the optimization of the problem will be very cumbersome, one possible solution is to solve the problem using simulation. Another practical extension of this study is to consider deteriorating products, for instance presented by Ullah et al. (2019b).

References

Accorsi, R., Cascini, A., Cholette, S., Manzini, R., and Mora, C. (2014). Economic and environmental assessment of reusable plastic containers: A food catering supply chain case study. *International Journal of Production Economics*, 152:88–101.

Ahmad, W., Sarkar, B., and Ullah, M. (2018). Impact of reparation for imperfect quality items having shortages in the system under multi-trade-credit-period. *DJ Journal of Engineering and Applied Mathematics*, 5:1–16.

Ahmadi, S. and Amin, S. H. (2019). An integrated chance-constrained stochastic model for a mobile phone closed-loop supply chain network with supplier selection. *Journal of Cleaner Production*, 226:988–1003.

Alegoz, M., Kaya, O., and Bayindir, Z. P. (2020). Closing the loop in supply chains: Economic and environmental effects. *Computers & Industrial Engineering*, 142:106366.

Almaraj, I. I. and Trafalis, T. B. (2019). An integrated multi-echelon robust closed-loop supply chain under imperfect quality production. *International Journal of Production Economics*, 218:212–227.

Amin, S. H. and Zhang, G. (2012). A proposed mathematical model for closed-loop network configuration based on product life cycle. *International Journal of Advanced Manufacturing Technology*, 58(5-8):791–801.

Asghar, I. and Kim, J. S. (2020). An automated smart epq-based inventory model for technology-dependent products under stochastic failure and repair rate. *Symmetry*, 12(3):388.

Asghar, I., Sarkar, B., and Kim, S. J. (2019). Economic analysis of an integrated production–inventory system under stochastic production capacity and energy consumption. *Energies*, 12(16):3179.

Assid, M., Gharbi, A., and Hajji, A. (2019). Production and setup control policy for unreliable hybrid manufacturing-remanufacturing systems. *Journal of Manufacturing Systems*, 50:103–118.

Bottani, E., Montanari, R., Rinaldi, M., and Vignali, G. (2015). Modeling and multi-objective optimization of closed loop supply chains: a case study. *Computers & Industrial Engineering*, 87:328–342.

Buchanan, D. and Abad, P. (1998). Optimal policy for a periodic review returnable inventory system. *IIE Transactions*, 30(11):1049–1055.

Cobb, B. R. (2016). Inventory control for returnable transport items in a closed-loop supply chain. *Transportation Research Part E: Logistics and Transportation Review*, 86:53–68.

Cui, Y. Y., Guan, Z., Saif, U., Zhang, L., Zhang, F., and Mirza, J. (2017). Close loop supply chain network problem with uncertainty in demand and returned products: Genetic artificial bee colony algorithm approach. *Journal of Cleaner Production*, 162:717–742.

Dai, Z. and Zheng, X. (2015). Design of close-loop supply chain network under uncertainty using hybrid genetic algorithm: A fuzzy and chance-constrained programming model. *Computers & Industrial Engineering*, 88:444–457.

Dhaiban, A. K., Baten, M. A., and Aziz, N. (2018). An optimal inventory control in hybrid manufacturing/remanufacturing system with deteriorating and defective items. *International Journal of Mathematics in Operational Research*, 12(1):66–90.

Dobos, I. and Richter, K. (2004). An extended production/recycling model with stationary demand and return rates. *International Journal of Production Economics*, 90(3):311–323.

Dobos, I. and Richter, K. (2006). A production/recycling model with quality consideration. *International Journal of Production Economics*, 104(2):571–579.

Dou, G., Guo, H., Zhang, Q., and Li, X. (2019). A two-period carbon tax regulation for manufacturing and remanufacturing production planning. *Computers & Industrial Engineering*, 128:502–513.

Dutta, P., Das, D., Schultmann, F., and Fröhling, M. (2016). Design and planning of a closed-loop supply chain with three way recovery and buy-back offer. *Journal of Cleaner Production*, 135:604–619.

El Saadany, A. M. and Jaber, M. Y. (2010). A production/remanufacturing inventory model with price and quality dependant return rate. *Computers & Industrial Engineering*, 58(3):352–362.

Fathi, M., Zandi, F., and Jouini, O. (2015). Modeling the merging capacity for two streams of product returns in remanufacturing systems. *Journal of Manufacturing Systems*, 37:265–276.

Giri, B. and Sharma, S. (2015). Optimizing a closed-loop supply chain with manufacturing defects and quality dependent return rate. *Journal of Manufacturing Systems*, 35:92–111.

Goh, T. and Varaprasad, N. (1986). A statistical methodology for the analysis of the life-cycle of reusable containers. *IIE Transactions*, 18(1):42–47.

Guo, J., He, L., and Gen, M. (2019). Optimal strategies for the closed-loop supply chain with the consideration of supply disruption and subsidy policy. *Computers & Industrial Engineering*, 128:886–893.

Guo, J. and Ya, G. (2015). Optimal strategies for manufacturing/remanufacturing system with the consideration of recycled products. *Computers & Industrial Engineering*, 89:226–234.

Habib, M. S., Asghar, O., Hussain, A., Imran, M., Mughal, M. P., and Sarkar, B. (2020). A robust possibilistic programming approach toward animal fat-based biodiesel supply chain network design under uncertain environment. *Journal of Cleaner Production*, page 122403.

Habib, M. S., Sarkar, B., Tayyab, M., Saleem, M. W., Hussain, A., Ullah, M., Omair, M., and Iqbal, M. W. (2019). Large-scale disaster waste management under uncertain environment. *Journal of Cleaner Production*, 212:200–222.

Hariga, M., As'ad, R., and Khan, Z. (2017). Manufacturing-remanufacturing policies for a centralized two stage supply chain under consignment stock partnership. *International Journal of Production Economics*, 183:362–374.

Hassanpour, A., Bagherinejad, J., and Bashiri, M. (2019). A robust leader-follower approach for closed loop supply chain network design considering returns quality levels. *Computers & Industrial Engineering*, 136:293–304.

Hekkert, M. P., Joosten, L. A., and Worrell, E. (2000). Reduction of co2 emissions by improved management of material and product use: the case of transport packaging. *Resources, Conservation and Recycling*, 30(1):1–27.

Heydari, J. and Ghasemi, M. (2018). A revenue sharing contract for reverse supply chain coordination under stochastic quality of returned products and uncertain remanufacturing capacity. *Journal of cleaner production*, 197:607–615.

Heydari, J., Govindan, K., and Sadeghi, R. (2018). Reverse supply chain coordination under stochastic remanufacturing capacity. *International Journal of Production Economics*, 202:1–11.

Hota, S. K., Sarkar, B., and Ghosh, S. K. (2020). Effects of unequal lot size and variable transportation in unreliable supply chain management. *Mathematics*, 8(3):357.

Iassinovskaia, G., Limbourg, S., and Riane, F. (2017). The inventory-routing problem of returnable transport items with time windows and simultaneous pickup and delivery in closed-loop supply chains. *International Journal of Production Economics*, 183:570–582.

İpekçi, A., Kam, M., and Saruhan, H. (2018). Investigation of 3d printing occupancy rates effect on mechanical properties and surface roughness of pet-g material products. *Journal of New Results in Science*, 7(2):1–8.

Iqbal, M. W. and Sarkar, B. (2019). Recycling of lifetime dependent deteriorated products through different supply chains. *RAIRO-Operations Research*, 53(1):129–156.

Joseph, O. and Sridharan, R. (2011). Evaluation of routing flexibility of a flexible manufacturing system using simulation modelling and analysis. *International Journal of Advanced Manufacturing Technology*, 56(1-4):273–289.

Kam, M., Saruhan, H., and İpekçi, A. (2018). Investigation the effects of 3d printer system vibrations on mechanical properties of the printed products. *Sigma J. Eng and Nat. Sci*, 36(3):655–666.

Kelle, P. and Silver, E. A. (1989). Forecasting the returns of reusable containers. *Journal of Operations Management*, 8(1):17–35.

Klumpp, M., Hesenius, M., Meyer, O., Ruiner, C., and Gruhn, V. (2019). Production logistics and human-computer interaction—state-of-the-art, challenges and requirements for the future. *International Journal of Advanced Manufacturing Technology*, 105(09):3691–3709.

Koh, S.-G., Hwang, H., Sohn, K.-I., and Ko, C.-S. (2002). An optimal ordering and recovery policy for reusable items. *Computers & Industrial Engineering*, 43(1-2):59–73.

Kuo, T.-C., Chang, S.-H., and Huang, S. H. (2006). Environmentally conscious design by using fuzzy multi-attribute decision-making. *International Journal of Advanced Manufacturing Technology*, 29(3-4):209–215.

Liao, H. and Deng, Q. (2018). Ees-eoq model with uncertain acquisition quantity and market demand in dedicated or combined remanufacturing systems. *Applied Mathematical Modelling*, 64:135–167.

Liao, H., Deng, Q., Wang, Y., Guo, S., and Ren, Q. (2018). An environmental benefits and costs assessment model for remanufacturing process under quality uncertainty. *Journal of Cleaner Production*, 178:45–58.

Liao, H. and Li, L. (2020). Environmental sustainability eoq model for closed-loop supply chain under market uncertainty: A case study of printer remanufacturing. *Computers & Industrial Engineering*, page 106525.

Liao, H., Shi, Y., Liu, X., Shen, N., and Deng, Q. (2019). A non-probabilistic model of carbon footprints in remanufacture under multiple uncertainties. *Journal of Cleaner Production*, 211:1127–1140.

Liu, C., Zhu, Q., Wei, F., Rao, W., Liu, J., Hu, J., and Cai, W. (2020). An integrated optimization control method for remanufacturing assembly system. *Journal of Cleaner Production*, 248:119261.

Liu, W., Ma, W., Hu, Y., Jin, M., Li, K., Chang, X., and Yu, X. (2019). Production planning for stochastic manufacturing/remanufacturing system with demand substitution using a hybrid ant colony system algorithm. *Journal of Cleaner Production*, 213:999–1010.

Liu, Z., Chen, J., and Diallo, C. (2018). Optimal production and pricing strategies for a remanufacturing firm. *International Journal of Production Economics*, 204:290–315.

Lu, M.-S. and Liu, Y.-J. (2011). Dynamic dispatching for a flexible manufacturing system based on fuzzy logic. *International Journal of Advanced Manufacturing Technology*, 54(9-12):1057–1065.

Lund, R. T. and Skeels, F. D. (1983). Guidelines for an original equipment manufacturer starting a remanufacturing operation. Technical report, Massachusetts Inst. of Tech., Cambridge (USA). Center for Policy Alternatives.

Mabini, M. C., Pintelon, L. M., and Gelders, L. F. (1992). Eoq type formulations for controlling repairable inventories. *International Journal of Production Economics*, 28(1):21–33.

Malik, A. I. and Sarkar, B. (2020). Coordination supply chain management under flexible manufacturing, stochastic leadtime demand, and mixture of inventory. *Mathematics*, 8(6):911.

Menderes, K., İpekçi, A., and Saruhan, H. (2017). Investigation of 3d printing filling structures effect on mechanical properties and surface roughness of pet-g material products. *Gaziosmanpaşa Bilimsel Araştırma Dergisi*, 6(Özel Sayı (ISMSIT2017)):114–121.

Menderes, K., Saruhan, H., and İpekçi, A. (2019). Investigation the effect of 3d printer system vibrations on surface roughness of the printed products. *Düzce Üniversitesi Bilim ve Teknoloji Dergisi*, 7(2):147–157.

Mishra, M., Hota, S. K., Ghosh, S. K., and Sarkar, B. (2020a). Controlling waste and carbon emission for a sustainable closed-loop supply chain management under a cap-and-trade strategy. *Mathematics*, 8(4):466.

Mishra, U., Wu, J.-Z., and Sarkar, B. (2020b). A sustainable production-inventory model for a controllable carbon emissions rate under shortages. *Journal of Cleaner Production*, 256:120268.

Mohammadi, M. (2020). Designing an integrated reliable model for stochastic lot-sizing and scheduling problem in hazardous materials supply chain under disruption and demand uncertainty. *Journal of Cleaner Production*, 274:122621.

Moshtagh, M. S. and Taleizadeh, A. A. (2017). Stochastic integrated manufacturing and remanufacturing model with shortage, rework and quality based return rate in a closed loop supply chain. *Journal of Cleaner Production*, 141:1548–1573.

Naeem, M. A., Dias, D. J., Tibrewal, R., Chang, P.-C., and Tiwari, M. K. (2013). Production planning optimization for manufacturing and remanufacturing system in stochastic environment. *Journal of Intelligent Manufacturing*, pages 1–12.

Nasr, N. (2019). *Remanufacturing in the Circular Economy: Operations, Engineering and Logistics*. John Wiley & Sons.

Niakan, F., Baboli, A., Moyaux, T., and Botta-Genoulaz, V. (2016). A new multi-objective mathematical model for dynamic cell formation under demand and cost uncertainty considering social criteria. *Applied Mathematical Modelling*, 40(4):2674–2691.

Onggo, B. S., Panadero, J., Corlu, C. G., and Juan, A. A. (2019). Agri-food supply chains with stochastic demands: A multi-period inventory routing problem with perishable products. *Simulation Modelling Practice and Theory*, 97:101970.

Ouaret, S., Kenné, J.-P., and Gharbi, A. (2018). Stochastic optimal control of random quality deteriorating hybrid manufacturing/remanufacturing systems. *Journal of Manufacturing Systems*, 49:172–185.

Piecyk, M. I. and McKinnon, A. C. (2010). Forecasting the carbon footprint of road freight transport in 2020. *International Journal of Production Economics*, 128(1):31–42.

Polotski, V., Kenne, J.-P., and Gharbi, A. (2015). Optimal production scheduling for hybrid manufacturing–remanufacturing systems with setups. *Journal of Manufacturing Systems*, 37:703–714.

Rahmani, K. and Yavari, M. (2019). Pricing policies for a dual-channel green supply chain under demand disruptions. *Computers & Industrial Engineering*, 127:493–510.

Richter, K. (1996). The eoq repair and waste disposal model with variable setup numbers. *European Journal of Operational Research*, 95(2):313–324.

Richter, K. (1997). Pure and mixed strategies for the eoq repair and waste disposal problem. *Operations-Research-Spektrum*, 19(2):123–129.

Richter, K. and Dobos, I. (1999). Analysis of the eoq repair and waste disposal problem with integer setup numbers. *International Journal of Production Economics*, 59(1-3):463–467.

Rokka, J. and Uusitalo, L. (2008). Preference for green packaging in consumer product choices–do consumers care? *International Journal of Consumer Studies*, 32(5):516–525.

Salema, M. I. G., Barbosa-Povoa, A. P., and Novais, A. Q. (2007). An optimization model for the design of a capacitated multi-product reverse logistics network with uncertainty. *European Journal of Operational Research*, 179(3):1063–1077.

Sarkar, B., Majumder, A., Sarkar, M., Kim, N., and Ullah, M. (2018). Effects of variable production rate on quality of products in a single-vendor multi-buyer supply chain management. *International Journal of Advanced Manufacturing Technology*, 99(1-4):567–581.

Sarkar, B., Shaw, B. K., Kim, T., Sarkar, M., and Shin, D. (2017a). An integrated inventory model with variable transportation cost, two-stage inspection, and defective items. *Journal of Industrial & Management Optimization*, 13(4):1975–1990.

Sarkar, B., Ullah, M., and Choi, S.-B. (2019). Joint inventory and pricing policy for an online to offline closed-loop supply chain model with random defective rate and returnable transport items. *Mathematics*, 7(6):497.

Sarkar, B., Ullah, M., and Kim, N. (2017b). Environmental and economic assessment of closed-loop supply chain with remanufacturing and returnable transport items. *Computers & Industrial Engineering*, 111:148–163.

Schrady, D. A. (1967). A deterministic inventory model for reparable items. *Naval Research Logistics Quarterly*, 14(3):391–398.

Shuang, Y., Diabat, A., and Liao, Y. (2019). A stochastic reverse logistics production routing model with emissions control policy selection. *International Journal of Production Economics*, 213:201–216.

Silva, D. A. L., Reno, G. W. S., Sevegnani, G., Sevegnani, T. B., and Truzzi, O. M. S. (2013). Comparison of disposable and returnable packaging: a case study of reverse logistics in brazil. *Journal of Cleaner Production*, 47:377–387.

Sundin, E. and Bras, B. (2005). Making functional sales environmentally and economically beneficial through product remanufacturing. *Journal of Cleaner Production*, 13(9):913–925.

Teunter, R. H. (1998). *Economic ordering quantities for remanufacturable item inventory systems.* Univ., Fak. für Wirtschaftswiss.

Tolooie, A., Maity, M., and Sinha, A. K. (2020). A two-stage stochastic mixed-integer program for reliable supply chain network design under uncertain disruptions and demand. *Computers & Industrial Engineering*, page 106722.

Ullah, M., Khan, I., and Sarkar, B. (2019a). Dynamic pricing in a multi-period newsvendor under stochastic price-dependent demand. *Mathematics*, 7(6):520.

Ullah, M. and Sarkar, B. (2018). Smart and sustainable supply chain management: A proposal to use rfid to improve electronic waste management. In Zhong R.Y., Dessouky M.I., X. X., editor, *Proceedings of International Conference on Computers and Industrial Engineering, CIE*, volume 2018-December.

Ullah, M. and Sarkar, B. (2020). Recovery-channel selection in a hybrid manufacturing-remanufacturing production model with rfid and product quality. *International Journal of Production Economics*, 219:360–374.

Ullah, M., Sarkar, B., and Asghar, I. (2019b). Effects of preservation technology investment on waste generation in a two-echelon supply chain model. *Mathematics*, 7(2):189.

Vahdani, B. and Ahmadzadeh, E. (2019). Designing a realistic ict closed loop supply chain network with integrated decisions under uncertain demand and lead time. *Knowledge-Based Systems*, 179:34–54.

Wang, J., Fei, Z., Chang, Q., Li, S., and Fu, Y. (2019). Multi-state decision of unreliable machines for energy-efficient production considering work-in-process inventory. *International Journal of Advanced Manufacturing Technology*, 102:1009–1021.

Wang, M. and Zhao, L. (2018). Pricing decisions and environmental assessment in a two-echelon supply chain with returnable transport items. *Procedia Computer Science*, 126:1792–1801.

Witt, C. E. (2000). Are reusable containers worth the cost? *Material Handling Management*, 55(7):75–80.

Zhang, Y., Alshraideh, H., and Diabat, A. (2018). A stochastic reverse logistics production routing model with environmental considerations. *Annals of Operations Research*, 271(2):1023–1044.

Zhang, Y. and Chen, W. (2020). Optimal production and financing portfolio strategies for a capital-constrained closed-loop supply chain with oem remanufacturing. *Journal of Cleaner Production*, page 123467.

A Appendix

A.1

Table 6: Input parameter values for numerical examples

P_m	100 $units/unit\ time$	h_f	3.2 $\$/unit/unit\ time$	h_u	1.2 $\$/unit/unit\ time$
S_m	100 $\$/setup$	O_m	20 $\$/order$	h_1	5.1 $\$/unit/unit\ time$
h_2	5.4 $\$/unit/unit\ time$	h_3	5.2 $\$/unit/unit\ time$	h_4	4.9 $\$/unit/unit\ time$
O_1	63 $\$/order$	C_{rc}	5 $\$/product$	C_τ	0.5 $\$/unit\ RTI$
h_R	1.5 $\$/unit/unit\ time$	s	2	k	4
C_q	1.5 $\$/product$	g_m	0.25 $gallon/unit\ distance$	a_1	4
$e_m\ kg/product$	0.01	C_{ec}	18 $\$/CO_2discharge$	C_g	0.02 $\$/product$
l_{1m}	373 $kilometers$	l_{2m}	226 $kilometers$	l_{3m}	216 $kilometers$
l_{4m}	371 $kilometers$	l_{1c}	373 $kilometers$	a_2	12
a_3	15	a_4	9	b_1	10
b_2	15	b_3	24	b_4	12
S_c	80 $\$/setup$	h_r	1.75 $\$/unit/unit\ time$	O_2	51 $\$/order$
O_3	49 $\$/order$	O_4	63 $\$/order$	γ	200
C_m	30 $\$/product$	C_t	0.04 $\$/container/unit\ distance$	q_f	0.8
q_u	0.2	C_r	15 $\$/RTIcapacity$	l_{2c}	226 $kilometers$
l_{3c}	216 $kilometers$	l_{4c}	371 $kilometers$	h_c	1.1 $\$/unit/unit\ time$

$t_2 = l_1 = \frac{\sum_{j=1}^{k} q_j}{Pm} + 0.009,$

$t_3 = l_2 = t_2 + 0.008$

$t_4 = l_3 = t_3 + 0.007$

$t_5 = l_4 = t_4 + 0.008$

A.2

The convexity of objective function Equation (29) is proved below; if,

$$a_{1,1} = \frac{\partial^2(\text{TCSC})}{\partial {q_1}^2},\ a_{1,2} = \frac{\partial^2(\text{TCSC})}{\partial\tau\partial q_1},\ a_{1,3} = \frac{\partial^2(\text{TCSC})}{\partial\theta\partial q_1},\ a_{1,4} = \frac{\partial^2(\text{TCSC})}{\partial q_2\partial q_1},\ a_{1,5} = \frac{\partial^2(\text{TCSC})}{\partial q_3\partial q_1},\ a_{1,6} = \frac{\partial^2(\text{TCSC})}{\partial q_4\partial q_1},$$

$$a_{2,1} = \frac{\partial^2(\text{TCSC})}{\partial q_1\partial\tau},\ a_{2,2} = \frac{\partial^2(\text{TCSC})}{\partial\tau^2},\ a_{2,3} = \frac{\partial^2(\text{TCSC})}{\partial\theta\partial\tau},\ a_{2,4} = \frac{\partial^2(\text{TCSC})}{\partial q_2\partial\tau},\ a_{2,5} = \frac{\partial^2(\text{TCSC})}{\partial q_3\partial\tau},\ a_{2,6} = \frac{\partial^2(\text{TCSC})}{\partial q_4\partial\tau},$$

$$a_{3,1} = \frac{\partial^2(\text{TCSC})}{\partial q_1\partial\theta},\ a_{3,2} = \frac{\partial^2(\text{TCSC})}{\partial\theta\partial\tau},\ a_{3,3} = \frac{\partial^2(\text{TCSC})}{\partial\theta^2},\ a_{3,4} = \frac{\partial^2(\text{TCSC})}{\partial q_2\partial\theta},\ a_{3,5} = \frac{\partial^2(\text{TCSC})}{\partial q_3\partial\theta},\ a_{3,6} = \frac{\partial^2(\text{TCSC})}{\partial q_4\partial\theta},$$

$$a_{4,1} = \frac{\partial^2(\text{TCSC})}{\partial q_1\partial q_2},\ a_{4,2} = \frac{\partial^2(\text{TCSC})}{\partial\tau\partial q_2},\ a_{4,3} = \frac{\partial^2(\text{TCSC})}{\partial\theta\partial q_2},\ a_{4,4} = \frac{\partial^2(\text{TCSC})}{\partial {q_2}^2},\ a_{4,5} = \frac{\partial^2(\text{TCSC})}{\partial q_3\partial q_2},\ a_{4,6} = \frac{\partial^2(\text{TCSC})}{\partial q_4\partial q_2},$$

$$a_{5,1} = \frac{\partial^2(\text{TCSC})}{\partial q_1\partial q_3},\ a_{5,2} = \frac{\partial^2(\text{TCSC})}{\partial\tau\partial q_3},\ a_{5,3} = \frac{\partial^2(\text{TCSC})}{\partial\theta\partial q_3},\ a_{5,4} = \frac{\partial^2(\text{TCSC})}{\partial q_2\partial q_3},\ a_{5,5} = \frac{\partial^2(\text{TCSC})}{\partial {q_3}^2},\ a_{5,6} = \frac{\partial^2(\text{TCSC})}{\partial q_4\partial q_3},$$

$$a_{6,1} = \frac{\partial^2(\text{TCSC})}{\partial q_1\partial q_4},\ a_{6,2} = \frac{\partial^2(\text{TCSC})}{\partial\tau\partial q_4},\ a_{6,3} = \frac{\partial^2(\text{TCSC})}{\partial\theta\partial q_4},\ a_{6,4} = \frac{\partial^2(\text{TCSC})}{\partial q_2\partial q_4},\ a_{6,5} = \frac{\partial^2(\text{TCSC})}{\partial q_3\partial q_4},\ a_{6,6} = \frac{\partial^2(\text{TCSC})}{\partial {q_4}^2}.$$

Then, for $k = 4$, the Hessian matrix of Equation (29) can be expressed as,

$$H = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} \\ a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} & a_{5,6} \\ a_{6,1} & a_{6,2} & a_{6,3} & a_{6,4} & a_{6,5} & a_{6,6} \end{pmatrix}$$

The Hessian matrix at q_1=5.76704, q_2=5.0737, q_3=10.689, q_4=4.28524, τ=6, θ=0.3328 is,

$$\begin{pmatrix} 14.0664 & -0.244055 & -13.2651 & -1.76557 & -1.02008 & -1.49912 \\ -0.244055 & 4.7566 & -11.5927 & 0.34871 & 0.258753 & -0.31349 \\ -13.2651 & -11.5927 & 776.205 & -25.769 & -18.7969 & -18.851 \\ -1.76557 & 0.34871 & -25.769 & 30.6247 & -1.97209 & -2.45113 \\ -1.02008 & 0.258753 & -18.7969 & -1.97209 & 9.17691 & -1.70564 \\ -1.49912 & -0.31349 & -18.851 & -2.45113 & -1.70564 & 31.3555 \end{pmatrix}$$

The first six principle minors are $+14.0664$, $+66.8487$, $+49085.9$, $+1.44387 \times 10^6$, $+1.2016 \times 10^7$, and $+3.52287 \times 10^8$. Hence, the expected cost is strictly convex at q_1=5.76704, q_2=5.0737, q_3=10.689, q_4=4.28524, τ=6, θ=0.3328.