ON THE DIOPHANTINE EQUATION $F_{n_1} + F_{n_2} + F_{n_3} = p_1^{z_1} \cdots p_s^{z_s}$

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ABSTRACT. Let F_n denote the n-th Fibonacci number and p_i the i-th prime number. In this paper we consider the Diophantine equation $F_{n_1} + F_{n_2} + F_{n_3} = p_1^{z_1} \cdots p_s^{z_s}$ in non-negative integers $n_1 \geq n_2 \geq n_3 \geq 0$ and non-negative integers z_i with $1 \leq i \leq s$. In particular, we completely solve the case that $s = 12$.

1. Introduction

There is a vast amount of literature on solving Diophantine equations involving binary recurrence sequences defined over the integers. Diophantine equations involving the sequence $(F_n)_{n\geq 0}$ of Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$ are one of the most frequently studied Diophantine equations involving binary sequences. In particular, Diophantine equations similar to the Diophantine equation posed in the title were studied by many authors. For instance, Bravo and Luca considered the case

$$u_n + u_m = 2^z,$$

where $(u_n)_{n\geq 0}$ is the Fibonacci sequence [6] and Lucas sequence [5] respectively. The case that $(u_n)_{n\geq 0}$ is the generalized k-Fibonacci sequence has been considered independently by Bravo et al. [4] and Marques [10].

In 2018, Pink and Ziegler [13] generalized the results due to Bravo and Luca [5, 6] by considering the Diophantine equation

$$u_n + u_m = wp_1^{z_1} \cdots p_s^{z_s} \tag{1}$$

in non-negative integer unknowns $n, m, z_1, \ldots, z_s$, where $(u_n)_{n\geq 0}$ is a binary non-degenerate recurrence sequence, $p_1, \ldots, p_s$ are distinct primes and w is a non-zero integer with $p_i \nmid w$ for $1 \leq i \leq s$. They proved that, under some mild condition to $(u_n)_{n\geq 0}$, there exists an effectively computable constant C depending only on $(u_n)_{n\geq 0}$, $w, s, p_1, \ldots, p_s$ such that all solutions $(n, m, z_1, \ldots, z_s)$ to equation (1) satisfy

$$\max\{n, m, z_1, \ldots, z_s\} < C.$$

Besides, they completely solved the equation

$$F_n + F_m = 2^{z_1} 3^{z_2} \cdots 199^{z_{46}} \tag{2}$$

and concluded that $\max\{n, m\} \leq 59$, and completely solved the equation

$$L_n + L_m = 2^{z_1} 3^{z_2} \cdots 199^{z_{46}},$$

where $(L_n)_{n\geq 0}$ is the Lucas sequence, and concluded that $\max\{n, m\} \leq 63$. These Diophantine equations are solved by the iterated application of linear forms in logarithms and LLL reduction algorithm. A novel step applied in [13] is the use of p-adic linear forms in logarithms, which helps obtaining a smaller first upper bound for the unknowns.

In this paper we consider a variant of equation (2). Again let F_n denote the n-th Fibonacci number and let p_i denote the i-th prime number. We would like to solve

$$F_{n_1} + F_{n_2} + F_{n_3} = p_1^{z_1} \cdots p_s^{z_s} \tag{3}$$

for non-negative integers $n_1 \geq n_2 \geq n_3$ and z_i with $1 \leq i \leq s$. In particular, we prove the following.

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Theorem 1. *Assume that* $(n_1, n_2, n_3, z_1, \dots, z_s)$ *is a solution to* (3). *Then*

$$n_1 < 2s^3(1.5\log s)^{2s}C(s)^2\left(\log(26s^3(1.5\log s)^{2s}C(s)^2)\right)^4,$$

where

$$C(s) = 18(s+3)!(s+2)^{s+3}(64)^{s+4}\log(4s+8).$$

In the case that $s = 12$ we solve Diophantine equation (3) completely. In particular, we obtain the following result.

Theorem 2. *In the case that* $s = 12$ *Diophantine equation* (3) *has exactly* 855 *solutions* $(n_1, n_2, n_3, z_1, \dots, z_{12})$. *In particular, there exists no solution with* $n_1 > 48$.

Let us note that in view of Theorem 2 we also have completely resolved the case that $s \leq 12$. Obviously all solutions to (3) for some $s \leq 12$ are contained in the set of solutions to (3) with $s = 12$. Therefore we can assume in the course of the proof of Theorems 1 and 2 that $s \geq 12$.

We will follow the ideas of [13] in our deduction of Theorems 1 and 2. The first key step is to obtain an (usually huge) absolute upper bound for n_1 and z_i with $1 \leq i \leq s$ by using the theory of linear forms in logarithms. The second key step is to reduce this first upper bound for n_1 and z_i with $1 \leq i \leq s$ by applying the LLL reduction method. The reduced upper bounds are small enough to find all solutions to (3) by a brute-force computer search.

2. Notations and technical lemmata

First, let us note that in view of Theorem 2 we may assume that $s \geq 12$. That is $P := \max_{1\leq i\leq s}\{p_i\} \geq 37$. During the proof we use the Binet formula for the Fibonacci sequence given the following form:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{for all } n \geq 0, \tag{4}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic polynomial $x^2 - x - 1$. Moreover, we have the inequality

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \qquad \text{for all } n \geq 1. \tag{5}$$

The following lemma gives upper and lower bounds for the quantity $F_{n_1} + F_{n_2} + F_{n_3}$, as well as upper and lower bounds for the exponents z_i in terms of n_1. These bounds will be utilized in the proof of Theorem 1 and Theorem 2.

Lemma 1. *The following holds:*

(i) We have

$$F_{n_1} + F_{n_2} + F_{n_3} \leq \frac{6}{1+\sqrt{5}}\alpha^{n_1}.$$

(ii) For all $n_1 \geq 1$, *we have*

$$z_i < \frac{1.1 n_1}{\log p_i}$$

for $i = 1, \dots, s$. *In particular, for all* $1 \leq i \leq s$ *we have* $z_i < 1.6 n_1$.

(iii) For all $n_1 \geq 1$, *we have*

$$0.38\alpha^{n_1} < F_{n_1} + F_{n_2} + F_{n_3}.$$

(iv) For all $n_1 \geq 1$, *we have*

$$n_1 < \frac{\sum_{i=1}^s z_i \log p_i}{\log \alpha} + 2.02.$$

Proof. We begin with the proof of *(i)*. Using (5) and noting that $n_1 \geq n_2 \geq n_3$, we have

$$F_{n_1} + F_{n_2} + F_{n_3} \leq 3F_{n_1} \leq 3\alpha^{n_1 - 1} = \frac{6}{1+\sqrt{5}}\alpha^{n_1}.$$

Next, we prove *(ii)*. By combining equation (3) and the just proved part *(i)* of the lemma we may write for every i with $1 \le i \le s$

$$p_i^{z_i} \le p_1^{z_1} \cdots p_s^{z_s} = F_{n_1} + F_{n_2} + F_{n_3} \le \frac{6}{1+\sqrt{5}} \alpha^{n_1}.$$

By taking logarithms to the above inequality, we obtain

$$z_i \log p_i \le n_1 \left(\log \alpha + \frac{1}{n_1} \log \left(\frac{6}{1+\sqrt{5}} \right) \right).$$

So whenever $n_1 \ge 1$, we have

$$z_i \le \frac{n_1}{\log p_i} \left(\log \alpha + \log \left(\frac{6}{1+\sqrt{5}} \right) \right) = \frac{n_1 \log 3}{\log p_i} < \frac{1.1 n_1}{\log p_i}.$$

In particular, for all $1 \le i \le s$ we have $z_i < \frac{1.1 n_1}{\log 2} < 1.6 n_1$.

We proceed to the proof of *(iii)*. Again, using (5), we have whenever $n_1 \ge 1$ that

$$\begin{aligned} F_{n_1} + F_{n_2} + F_{n_3} &\ge \frac{1}{\alpha^2} \left(\alpha^{n_1} + \alpha^{n_2} + \alpha^{n_3} \right) \\ &= \frac{\alpha^{n_1}}{\alpha^2} \left(1 + \frac{1}{\alpha^{n_1 - n_2}} + \frac{1}{\alpha^{n_1 - n_3}} \right) \\ &> \frac{\alpha^{n_1}}{\alpha^2} > 0.38 \alpha^{n_1}. \end{aligned}$$

Finally, we prove *(iv)*. We combine the inequality from *(iii)* with equation (3) to get

$$0.38 \alpha^{n_1} < F_{n_1} + F_{n_2} + F_{n_3} = p_1^{z_1} \cdots p_s^{z_s},$$

and by taking logarithms this yields

$$n_1 \log \alpha < -\log(0.38) + \sum_{i=1}^{s} z_i \log p_i$$

Thus, whenever $n_1 \ge 1$, we have

$$n_1 < \frac{\sum_{i=1}^{s} z_i \log p_i}{\log \alpha} - \frac{\log(0.38)}{\log \alpha} < \frac{\sum_{i=1}^{s} z_i \log p_i}{\log \alpha} + 2.02.$$

□

In order to estimate the n-th prime number we will use the following lemma which is derived using the results due to Rosser [14].

Lemma 2. *The following holds:*

(i) For $n \ge 3$, we have

$$p_n < 1.74 n \log n.$$

(ii) For $n \ge 12$, we have

$$\log p_n < 1.5 \log n.$$

Proof. We begin with the proof of *(i)*. It is trivially true for $n = 3$. For $n > 3$, we have $p_n < n(\log n + 2 \log \log n)$ from [14] so that

$$p_n < n(\log n + 2 \log \log n) = n \log n \left(1 + \frac{2 \log \log n}{\log n} \right) < 1.74 n \log n.$$

Next, we prove *(ii)*. By [14], we have $p_n < n(\log n + \log \log n)$ for $n \ge 6$ so that

$$\log p_n < \log n + \log(\log n + \log \log n) = \log n \left(1 + \frac{\log(\log n + \log \log n)}{\log n} \right) < 1.5 \log n$$

for $n \ge 12$. □

For the third technical lemma, we state an elementary result due to Pethő and de Weger [12]. It will be frequently used in the proof of Theorem 1. For a proof of Lemma 3 we refer to [15, Appendix B].

Lemma 3. *Let u, $v \geq 0$, $h \geq 1$ and $x \in \mathbb{R}$ be the largest solution of $x = u + v(\log x)^h$. Then*
$$x < \max\left\{2^h(u^{1/h} + v^{1/h}\log(h^h v))^h, 2^h(u^{1/h} + 2e^2)^h\right\}.$$

In view of the use of two results on linear forms in logarithms in Section 3 and Section 4, let us introduce some further notations. Let L be a number field and $\eta \in L$. We denote by $h(\eta)$ as usual the *absolute logarithmic Weil height* of η, that is
$$h(\eta) = \frac{1}{d}\left(\log|a_0| + \sum_{i=1}^{d}\log\max\left(|\eta^{(i)}|, 1\right)\right),$$
where a_0 and d are respectively the leading coefficient and the degree of the minimal polynomial of η over $\mathbb{Z}$, and the $\eta^{(i)}$ are the conjugates of η in $\mathbb{C}$. We will use the following well known properties of the absolute logarithmic height for any $\eta, \gamma \in \overline{\mathbb{Q}}$ and $l \in \mathbb{Q}$ (see for example [20, Chapter 3] for a detailed reference):
$$\begin{aligned} h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2, \\ h(\eta\gamma^{\pm 1}) &\leq h(\eta) + h(\gamma), \\ h(\eta^l) &= |l| h(\eta). \end{aligned}$$

3. Application of p-adic methods

In this section we prove an upper bound for n_1 in terms of $n_1 - n_3$ and $\log n_1$. To achieve this, we apply the results of Bugeaud and Laurent [7] in linear forms in two p-adic logarithms. Their result yields an upper bound of the p-adic valuation involving two algebraic numbers. It requires the assumption that the algebraic numbers involved in the p-adic logarithms are multiplicatively independent, in contrast to the results due to Yu [18, 19] which can be applied without this assumption. Nevertheless we use the result due to Bugeaud and Laurent [7] since their result yields rather small upper bounds. We shall deduce the upper bound of z_i (in terms of $n_1 - n_3$ and $\log n_1$) for the case when the algebraic numbers are multiplicatively dependent separately, and show that the bound is smaller than that obtained when the algebraic numbers are multiplicatively independent. Then by applying Lemma 1 *(iv)* we obtain an upper bound for n_1 in terms of $n_1 - n_3$ and $\log n_1$. To be self-contained, we begin with introducing some notations.

For a prime number p denote by $\mathbb{Q}_p$ the field of p-adic numbers with the standard p-adic valuation ord_p. Denote by $\overline{\mathbb{Q}}_p$ an algebraic closure of the p-adic field $\mathbb{Q}_p$. We equip the field $\overline{\mathbb{Q}}_p$ with the ultrametric absolute value $|x|_p = p^{-v_p(x)}$, where v_p denotes the unique extension to $\overline{\mathbb{Q}}_p$ of the standard p-adic valuation ord_p over $\mathbb{Q}_p$ normalized by $v_p(p) = 1$ (we set $v_p(0) = +\infty$). Let α_1, α_2 be algebraic numbers over $\mathbb{Q}$ and we regard them as elements of the field $K = \mathbb{Q}(\alpha_1, \alpha_2) \subseteq L = \mathbb{Q}_p(\alpha_1, \alpha_2) \subset \overline{\mathbb{Q}}_p$. Note, that for every non-zero $\delta \in L$ we have
$$v_p(\delta) = \frac{\mathrm{ord}_p(\mathrm{N}_{L/\mathbb{Q}_p}(\delta))}{d_L},$$
and in particular for $\delta \in K$ we have
$$v_p(\delta) = \frac{\mathrm{ord}_p(\mathrm{N}_{K/\mathbb{Q}}(\delta))}{[K:\mathbb{Q}]}, \tag{6}$$
where $d_L = [L : \mathbb{Q}_p]$ is the degree of the field extension $L/\mathbb{Q}_p$ and $\mathrm{N}_{L/\mathbb{Q}_p}(\delta)$ is the norm of δ with respect to $\mathbb{Q}_p$. Denote by e the ramification index of the local field extension $L/\mathbb{Q}_p$ and by f the residual degree of this extension. Put
$$\mathcal{D} = \frac{[\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]}{f}.$$
Let $A_1 > 1$, $A_2 > 1$ be two real numbers such that
$$\log A_i \geq \max\left\{h(\alpha_i), \frac{\log p}{\mathcal{D}}\right\}, \qquad (i = 1, 2).$$

Let $\mathcal{K}$ be a complete field containing $\mathbb{Q}_p$ whose absolute value $|\cdot|_p$ extends that of $\mathbb{Q}_p$. The formal power series for the logarithm is

$$\mathrm{Log}(1+Z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} Z^n}{n}.$$

It has the region of convergence $|Z|_p < 1$. We define the p-adic logarithmic function of $z \in \mathcal{K}$ with $|z-1|_p < 1$ as

$$\log_p z = \log_p(1 + (z-1)) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(z-1)^n}{n}.$$

We note that for any $x, y \in \mathcal{K}$ with $|x-1|_p < 1$ and $|y-1|_p < 1$, we have $\log_p(xy) = \log_p(x) + \log_p(y)$, and for every $\xi \in \mathcal{K}$ with $|\xi - 1|_p < p^{-1/(p-1)}$ we have

$$|\log_p \xi|_p = |\xi - 1|_p, \tag{7}$$

or equivalently

$$v_p(\log_p \xi) = v_p(\xi - 1) \tag{8}$$

provided that $|\xi - 1|_p < p^{-1/(p-1)}$. Note that in the case that $|\xi - 1|_p = p^{-1/(p-1)}$ we can replace in equation (7) the $=$ sign by a $\leq$ sign (or equivalently in equation (8) the $=$ sign by a $\geq$ sign).

We now state the result due to Bugeaud and Laurent [7, Corollary 1]:

Theorem 3. *Let b_1, b_2 be positive integers and suppose that α_1 and α_2 are multiplicatively independent algebraic numbers such that $v_p(\alpha_1) = v_p(\alpha_2) = 0$. Put*

$$b' := \frac{b_1}{\mathcal{D} \log A_2} + \frac{b_2}{\mathcal{D} \log A_1}. \tag{9}$$

Then we have

$$v_p(\alpha_1^{b_1}\alpha_2^{b_2} - 1) \leq \frac{24p(p^f - 1)\mathcal{D}^4}{(p-1)(\log p)^4} B^2 \log A_1 \log A_2, \tag{10}$$

with

$$B := \max\left\{\log b' + \log\log p + 0.4, 10, \frac{10\log p}{\mathcal{D}}\right\}. \tag{11}$$

We shall apply the above result in the case where $L = \mathbb{Q}_p(\sqrt{5})$, that is, $d_L \leq 2$ and $1 \leq \mathcal{D}, e, f \leq 2$. It should be noted that, we have $e = 2$ if and only if $p = 5$ and $f = 2$ if and only if $\left(\frac{5}{p}\right) = -1$, where $\left(\frac{5}{p}\right)$ denotes the Legendre symbol. We obtain Corollary 1 below.

Corollary 1. *Let $B' = \max\left\{b_1, b_2, p^{10}, e^{10}\right\}$ and with the notations and assumptions of Theorem 3 in force we have*

$$v_p(\alpha_1^{b_1}\alpha_2^{b_2} - 1) \leq C_1(p) \log A_1 \log A_2 (\log B')^2,$$

where

$$C_1(p) = \frac{947p^f}{(\log p)^4}.$$

Proof. First, let us note that since $f, \mathcal{D} \leq 2$ one easily deduces that

$$\frac{24p(p^f-1)\mathcal{D}^4}{(p-1)(\log p)^4} < \frac{768p^f}{(\log p)^4}.$$

Moreover,

$$b' := \frac{b_1}{\mathcal{D}\log A_2} + \frac{b_2}{\mathcal{D}\log A_1} \leq \frac{b_1 + b_2}{\log p} \leq \frac{2\max\{b_1, b_2\}}{\log p}$$

and since we assume that $B' \geq \max\{p^{10}, e^{10}\}$ we get

$$B = \max\left\{\log b' + \log\log p + 0.4, 10, \frac{10\log p}{\mathcal{D}}\right\}$$
$$\leq \log(2e^{0.4}B') = \left(\frac{\log(2e^{0.4})}{\log B'} + 1\right)\log B' < 1.11\log B'.$$

Putting everything together yields the corollary. □

We also want to find bounds for the p-adic valuation of $\left(\frac{\beta}{\alpha}\right)^n - 1$.

Lemma 4. *Let p be a prime and write*

$$u = v_p\left(\left(\frac{\beta}{\alpha}\right)^{p^f-1} - 1\right).$$

Then

$$v_p\left(\left(\frac{\beta}{\alpha}\right)^n - 1\right) \leq v_p(n) + u \leq \frac{\log n}{\log p} + \frac{(p^2-1)\log|\alpha/\beta|}{\log p}.$$

Proof. The proof in the case that $p = 2$ is given in detail in [16]. The result in the case that p is odd is due to the fact that the multiplicative order m of β/α modulo p, divides $p^f - 1$ and therefore we have

$$v_p\left((\beta/\alpha)^m - 1\right) \leq v_p\left((\beta/\alpha)^{p^f-1} - 1\right) = u.$$

Since for any algebraic integer $x \in \mathbb{Q}(\sqrt{5})$ and any odd prime p with $v_p(x-1) > 0$ we have $v_p(x-1) > 1/(p-1)$ we get

$$v_p(x^p - 1) = v_p(x-1) + 1.$$

Thus we obtain the lemma also in the case that p is odd.

We are left to find an upper bound for u. Since $f = 1, 2$ we obtain

$$v_p\left((\beta/\alpha)^{p^f-1} - 1\right) = \frac{\mathrm{ord}_p(\mathrm{N}_{K/\mathbb{Q}}((\beta/\alpha)^{p^2-1} - 1))}{2}$$
$$\leq \frac{\log\left|\left((\beta/\alpha)^{p^2-1} - 1\right)\left((\alpha/\beta)^{p^2-1} - 1\right)\right|}{2\log p}$$
$$\leq \frac{\log\left|(\alpha/\beta)^{p^2-1} - 1\right|}{\log p} \leq \frac{(p^2-1)\log|\alpha/\beta|}{\log p}.$$

Note that since p is odd we have $0 < 1 - (\beta/\alpha)^{p^2-1} < 1$. □

Next, we want to find an upper bound for z_i, with $i = 1, \ldots, s$ in terms of $n_1 - n_3$ and $\log n_1$. Let us fix the index i. To simplify the notation, in the remaining part of the proof we write $p = p_i$ and so on. Further, we will work in the field

$$K = \mathbb{Q}(\alpha, \beta) = \mathbb{Q}(\sqrt{5}) \subseteq L = \mathbb{Q}_p(\alpha, \beta).$$

Note that we have $v_p(\alpha) = 0$ and $v_p(\beta) = 0$.

Let us consider equation (3) and rewrite it as

$$\Lambda = \frac{\beta^{n_1-n_3} + \beta^{n_2-n_3} + 1}{\alpha^{n_1-n_3} + \alpha^{n_2-n_3} + 1}\left(\frac{\beta}{\alpha}\right)^{n_3} - 1 = \frac{-(\alpha-\beta)p_1^{z_1}\cdots p_s^{z_s}}{\alpha^{n_3}(\alpha^{n_1-n_3} + \alpha^{n_2-n_3} + 1)}. \tag{12}$$

Note that in the case that $n_1 = n_3$, (12) turns into

$$\Lambda = \left(\frac{\beta}{\alpha}\right)^{n_1} - 1 = \frac{-(\alpha-\beta)p_1^{z_1}\cdots p_s^{z_s}}{3\alpha^{n_1}}.$$

Let us note that $\alpha - \beta = \sqrt{5}$ is an algebraic integer, hence $v_p(\alpha - \beta) \geq 0$. Thus, the above immediately implies

$$v_p(\Lambda) \geq z. \tag{13}$$

Applying Lemma 4 we obtain in this case

$$z \le \frac{\log n_1}{\log p} + \frac{(p^2-1)\log|\alpha/\beta|}{\log p} < \frac{p^2 \log n_1}{\log p}$$

provided that $n_1 \ge 3$. If we combine this inequality with Lemma 1 *(iv)* and Lemma 2 *(i)* we obtain

$$\begin{aligned} n_1 &< \frac{\left(\sum_{i=1}^s p_i^2\right)\log n_1}{\log\alpha} + 2.02 < \frac{s p_s^2 \log n_1}{\log \alpha} + 2.02 < \frac{s(1.74 s\log s)^2 \log n_1}{\log\alpha} + 2.02 \\ &< 6.292 s^3 (\log s)^2 \log n_1 + 2.02 < 6.3 s^3 (\log s)^2 \log n_1 \end{aligned}$$

where $s \ge 12$ and $n_1 \ge 3$. However an application of Lemma 3 yields

$$n_1 < 12.6 s^3 (\log s)^2 \log(6.3 s^3 (\log s)^2)$$

which is much smaller than the upper bound given in Theorem 1. We also note that in the case that $s = 12$ we obtain $n_1 < 1.5 \times 10^6$ and this bound is also much smaller than what we will obtain later in Proposition 2. Therefore we may assume that $n_1 > n_3$.

We proceed to compute $v_p(\Lambda)$ by considering the right hand side of equation (12) and we obtain

$$v_p(\Lambda) = z - v_p(\alpha^{n_1-n_3} + \alpha^{n_2-n_3} + 1) + v_p(\alpha - \beta).$$

Let us note again that $v_p(\alpha-\beta) \ge 0$. Moreover, using formula (6) we have

$$\begin{aligned} v_p(\alpha^{n_1-n_3} + \alpha^{n_2-n_3} + 1) &= \frac{\mathrm{ord}_p(\mathrm{N}_{K/\mathbb{Q}}(\alpha^{n_1-n_3} + \alpha^{n_2-n_3} + 1))}{[K:\mathbb{Q}]} \\ &= \frac{\mathrm{ord}_p((\alpha^{n_1-n_3} + \alpha^{n_2-n_3} + 1)(\beta^{n_1-n_3} + \beta^{n_2-n_3} + 1))}{2} \\ &\le \frac{\log|(\alpha^{n_1-n_3} + \alpha^{n_2-n_3} + 1)(\beta^{n_1-n_3} + \beta^{n_2-n_3} + 1)|}{2\log p} \\ &\le \frac{\log(3|\alpha^{n_1-n_3} + \alpha^{n_2-n_3} + 1|)}{\log p} \\ &\le (n_1-n_3)\frac{\log(9\alpha)}{\log p}. \end{aligned}$$

and we obtain

$$v_p(\Lambda) \ge z - (n_1 - n_3)\frac{\log(9\alpha)}{\log p}. \tag{14}$$

We shall apply Corollary 1 in order to bound $v_p(\Lambda)$ from above. Now, we estimate the heights of $\eta_1 = \frac{\beta^{n_1-n_3}+\beta^{n_2-n_3}+1}{\alpha^{n_1-n_3}+\alpha^{n_2-n_3}+1}$ and $\eta_2 = \frac{\beta}{\alpha}$.

First, note that $\frac{\beta^{n_1-n_3}+\beta^{n_2-n_3}+1}{\alpha^{n_1-n_3}+\alpha^{n_2-n_3}+1}$ and $\frac{\alpha^{n_1-n_3}+\alpha^{n_2-n_3}+1}{\beta^{n_1-n_3}+\beta^{n_2-n_3}+1}$ are conjugates of a degree two polynomial. Referring to the argument from [13], we have

$$\begin{aligned} h\left(\frac{\beta^{n_1-n_3}+\beta^{n_2-n_3}+1}{\alpha^{n_1-n_3}+\alpha^{n_2-n_3}+1}\right) &\le \max\{\log|\beta^{n_1-n_3}+\beta^{n_2-n_3}+1|, \log|\alpha^{n_1-n_3}+\alpha^{n_2-n_3}+1|\} \\ &= \log|\alpha^{n_1-n_3}+\alpha^{n_2-n_3}+1| \\ &= \log 3 + (n_1-n_3)\log\alpha \\ &\le (n_1-n_3)\log(3\alpha). \end{aligned} \tag{15}$$

Hence we can take

$$\log A_1 = \max\{(n_1-n_3)\log(3\alpha), \log p\},$$

so that

$$\log A_1 \ge \max\left\{h\left(\frac{\beta^{n_1-n_3}+\beta^{n_2-n_3}+1}{\alpha^{n_1-n_3}+\alpha^{n_2-n_3}+1}\right), \frac{\log p}{\mathcal{D}}\right\}.$$

If we assume that $n_1 - n_3 > \log P$, where $P := \max_{1\le i\le s}\{p_i\}$ as defined in Section 2, we have $\log A_1 = (n_1-n_3)\log(3\alpha)$.

Next, we note that $\frac{\beta}{\alpha} = -\beta^2$, hence

$$h\left(\frac{\beta}{\alpha}\right) = h(\beta^2) = 2h(\beta) = \log\left(\max\{|\beta|, 1\}\right) + \log\left(\max\{|\alpha|, 1\}\right) = \log\alpha < 0.5$$

and we can take

$$\log A_2 = \max\{0.5, \log p\} = \log p \geq \max\left\{h\left(\frac{\beta}{\alpha}\right), \frac{\log p}{\mathcal{D}}\right\}.$$

We shall now distinguish between two cases, namely whether $\frac{\beta^{n_1-n_3}+\beta^{n_2-n_3}+1}{\alpha^{n_1-n_3}+\alpha^{n_2-n_3}+1}$ and $\frac{\beta}{\alpha}$ are multiplicatively independent or not.

Case I. In this case we assume that $\frac{\beta^{n_1-n_3}+\beta^{n_2-n_3}+1}{\alpha^{n_1-n_3}+\alpha^{n_2-n_3}+1}$ and $\frac{\beta}{\alpha}$ are multiplicatively dependent. Then there exist coprime integers r and t such that

$$\frac{\beta^{n_1-n_3}+\beta^{n_2-n_3}+1}{\alpha^{n_1-n_3}+\alpha^{n_2-n_3}+1} = \left(\frac{\alpha}{\beta}\right)^{r/t}.$$

First, we note that $\delta := (\frac{\alpha}{\beta})^{1/t} = (-\alpha^2)^{1/t}$ has to be still of degree two. Indeed $(\frac{\alpha}{\beta})^{r/t} \in K$ therefore $(\frac{\alpha}{\beta})^r \in K^t$. Since r and t are coprime by assumption, we obtain $(\frac{\alpha}{\beta}) \in K^t$. Hence $\delta := (\frac{\alpha}{\beta})^{1/t} \in K$. Since α is a fundamental unit and since $-\alpha^2 < 0$ we deduce that $t = 1$. From the upper bound (15) we find that

$$(n_1 - n_3)\log(3\alpha) \geq h\left(\frac{\beta^{n_1-n_3}+\beta^{n_2-n_3}+1}{\alpha^{n_1-n_3}+\alpha^{n_2-n_3}+1}\right) = |r| h(\alpha/\beta) = |r|\log\alpha,$$

that is $|r| < 3.3(n_1 - n_3)$. In particular, we have

$$-n_3 t + r \leq n_3 + |r| < 3.3\, n_1. \tag{16}$$

Therefore we obtain from (14)

$$v_p(\Lambda) = v_p(\delta^{-n_3 t + r} - 1) \geq z - \frac{\log(9\alpha)}{\log p}(n_1 - n_3).$$

If we suppose that $z \geq \frac{3}{2} + \frac{\log(9\alpha)}{\log p}(n_1 - n_3)$, we have $v_p(\Lambda) = v_p(\delta^{-n_3 t+r} - 1) \geq \frac{3}{2} > \frac{1}{p-1}$. We may use property (8) of the p-adic logarithm to obtain

$$v_p(\log_p \delta) + v_p(-n_3 t + r) = v_p(\log_p \delta^{-n_3 t + r}) = v_p(\delta^{-n_3 t + r} - 1) \geq z - \frac{\log(9\alpha)}{\log p}(n_1 - n_3). \tag{17}$$

Note that

$$v_p\left(\log_p(\alpha/\beta)\right) = v_p\left(\frac{\log_p\left((\alpha/\beta)^{p^2-1}\right)}{p^2-1}\right) = v_p\left((\alpha/\beta)^{p^2-1} - 1\right)$$

holds for odd p. Indeed the equality holds, since $v_p(\log_p(x)) = v_p(x-1)$ if $v_p(x-1) > 1/(p-1)$ which certainly holds for odd p and $x \in \mathbb{Q}(\sqrt{5})$ provided that $v_p(x-1) > 0$. Therefore we obtain in the case that p is odd

$$\begin{aligned} v_p(\log_p \delta) &= v_p\left(\log_p(\alpha/\beta)\right) = v_p((\alpha/\beta)^{p^2-1} - 1) \\ &= \mathrm{ord}_p\left(((\alpha/\beta)^{p^2-1} - 1)((\beta/\alpha)^{p^2-1} - 1)\right) \\ &\leq \frac{\log\left|2(\alpha/\beta)^{p^2-1}\right|}{\log p} \\ &\leq \frac{(p^2-1)\log|2\alpha/\beta|}{2\log p} < \frac{0.83p^2}{\log p}. \end{aligned}$$

In the case that $p = 2$ we obtain by a direct computation that

$$v_2(\log_2 \delta) = v_2\left(\log_2(\alpha/\beta)\right) = 2 < \frac{0.83p^2}{\log p}.$$

If we assume that $n_1 > 3$ we get, by using (16) and (17), a very crude estimate

$$z < v_p(\log_p \delta) + \frac{2\log n_1}{\log p} + \frac{\log(9\alpha)}{\log p}(n_1 - n_3) < \frac{4p^2}{\log p}(n_1 - n_3)\log n_1. \tag{18}$$

We note that $z < \frac{3}{2} + \frac{\log(9\alpha)}{\log p}(n_1 - n_3)$ is already contained in (18) for all $i = 1, \dots, s$.

Case II. In this case we assume that $\frac{\beta^{n_1-n_3}+\beta^{n_2-n_3}+1}{\alpha^{n_1-n_3}+\alpha^{n_2-n_3}+1}$ and $\frac{\beta}{\alpha}$ are multiplicatively independent. We note that α and β (respectively $\alpha^{n_1-n_3} + \alpha^{n_2-n_3} + 1$ and $\beta^{n_1-n_3} + \beta^{n_2-n_3} + 1$) are conjugates of a degree two polynomial. Hence, $v_p(\alpha) = v_p(\beta)$ (respectively $v_p(\alpha^{n_1-n_3} + \alpha^{n_2-n_3} + 1) = v_p(\beta^{n_1-n_3} + \beta^{n_2-n_3} + 1)$) so that

$$v_p\left(\frac{\beta^{n_1-n_3} + \beta^{n_2-n_3} + 1}{\alpha^{n_1-n_3} + \alpha^{n_2-n_3} + 1}\right) = v_p\left(\frac{\beta}{\alpha}\right) = 0. \tag{19}$$

Now, we apply Corollary 1 to obtain

$$v_p(\Lambda) \le C_1(p)\log(3\alpha)(n_1 - n_3)\log p\,(\max\{\log n_1, 10\log p, 10\})^2. \tag{20}$$

If we assume that $n_1 > P^{10}$, we have $\max\{\log n_1, 10\log p, 10\} = \log n_1$. Comparing upper and lower bounds of $v_p(\Lambda)$ from inequalities (14) and (20) yields

$$C_1(p)\log(3\alpha)(n_1 - n_3)\log p\,(\log n_1)^2 \ge z - \frac{\log(9\alpha)}{\log p}(n_1 - n_3).$$

Thus we obtain the upper bound

$$\begin{aligned} z &\le C_1(p)\log(3\alpha)(n_1 - n_3)\log p\,(\log n_1)^2 + \frac{\log(9\alpha)}{\log p}(n_1 - n_3) \\ &< \left(C_1(p)\log p\log(3\alpha) + \frac{\log(9\alpha)}{\log p}\right)(n_1 - n_3)(\log n_1)^2. \end{aligned}$$

Comparing the results from Case I and Case II under the assumptions that $n_1 > P^{10}$ and $n_1 - n_3 > \log P$, we note that the bound from Case II is always larger than the bound from Case I. Therefore we have the overall upper bound

$$z \le \left(C_1(p)\log p\log(3\alpha) + \frac{\log(9\alpha)}{\log p}\right)(n_1 - n_3)(\log n_1)^2 < \frac{1500p^2}{(\log p)^3}(n_1 - n_3)(\log n_1)^2. \tag{21}$$

This upper bound combined with Lemma 1 yields in the case that $s = 12$ the upper bound

$$\begin{aligned} n_1 &\le \sum_{i=1}^{12} \frac{z_i \log p_i}{\log \alpha} + 2.02 < \frac{1500}{\log \alpha}\sum_{i=1}^{12}\frac{p_i^2}{(\log p_i)^2}(n_1 - n_3)(\log n_1)^2 + 2.02 \\ &< 1.5 \times 10^6 (n_1 - n_3)(\log n_1)^2 \end{aligned} \tag{22}$$

provided that $n_1 > 4.81 \times 10^{15} > 37^{10}$ and $n_1 - n_3 \ge 4 > \log(37)$.

Let us consider the general case. First, we note that due to Lemma 2 *(i)* we have $\frac{p_n^2}{(\log p_n)^2} < \frac{p_n^2}{(\log n)^2} < (1.74n)^2 < 3.03s^2$ for $3 \le n \le s$. An application of Lemma 1 yields

$$\begin{aligned} n_1 &\le \sum_{i=1}^{s} \frac{z_i \log p_i}{\log \alpha} + 2.02 < \frac{1500}{\log \alpha}\sum_{i=1}^{s}\frac{p_i^2}{(\log p_i)^2}(n_1 - n_3)(\log n_1)^2 + 2.02 \\ &= \frac{1500}{\log \alpha}\sum_{i=3}^{s}\frac{p_i^2}{(\log p_i)^2}(n_1 - n_3)(\log n_1)^2 + \frac{1500}{\log \alpha}\sum_{i=1}^{2}\frac{p_i^2}{(\log p_i)^2}(n_1 - n_3)(\log n_1)^2 + 2.02 \\ &< \frac{1500(3.03)}{\log \alpha}s^2(s-2)(n_1 - n_3)(\log n_1)^2 + 50000(n_1 - n_3)(\log n_1)^2 + 2.02 \\ &< 9500s^3(n_1 - n_3)(\log n_1)^2 \end{aligned}$$

provided that $n_1 > P^{10}$ and $n_1 - n_3 > \log P$, where $P = \max_{1 \le i \le s}\{p_i\}$.

Let us briefly discuss the case that $n_1 - n_3 \leq \log P$. In this case we can choose $\log A_1 = (\log P)\log(3\alpha)$. Then replacing $n_1 - n_3$ by $\log P$ in the calculations that lead to Proposition 1 and using $\log P < 1.5 \log s$ due to Lemma 2 *(ii)* lead to the inequality

$$n_1 < 9500 s^3 (\log P)(\log n_1)^2 < 15000 s^3 (\log s)(\log n_1)^2$$

on assuming $n_1 > P^{10}$. An application of Lemma 3 yields an upper bound for n_1 which is smaller than the upper bound given in Theorem 1.

In the case that $s = 12$ we replace $n_1 - n_3$ by $\log P$ in (22) to obtain the inequality $n_1 < 1.5 \times 10^6 \log P (\log n_1)^2$, giving

$$n_1 < 5.5 \times 10^6 (\log n_1)^2,$$

which implies $n_1 < 6.3 \times 10^9$ by Lemma 3. However, it contradicts the assumption that $n_1 > P^{10} = 37^{10}$. Thus, we have $n_1 \leq 37^{10} < 4.81 \times 10^{15}$.

Let us summarize the results found so far:

Proposition 1. *Assume that $s \geq 12$ and that $n_1 > P^{10}$ and that $n_1 - n_3 > \log P$, then we have*

$$n_1 < 9500 s^3 (n_1 - n_3)(\log n_1)^2.$$

Moreover, in the case that $s = 12$ we have

$$n_1 < 1.5 \times 10^6 (n_1 - n_3)(\log n_1)^2,$$

provided that $n_1 > 4.81 \times 10^{15}$.

4. Application of linear forms in complex logarithms

The main purpose of this section is to find explicit upper bounds for n_1 and the exponents z_i with $1 \leq i \leq s$. We assume that $n_1 \geq n_2 \geq n_3$, $n_1 - n_3 > \log P$ and $n_1 > P^{10}$ hold so that we can apply Proposition 1 whenever necessary.

In the following we apply lower bounds for linear forms in complex logarithms. In particular, we shall apply the following theorem by Baker and Wüstholz (see [2] for a more precise statement).

Theorem 4. *For a linear form $\Lambda = b_1 \log \eta_1 + \cdots + b_k \log \eta_k \neq 0$ in logarithms of k algebraic numbers $\eta_1, \ldots, \eta_k$ with rational integer coefficients $b_1, \ldots, b_k$, we have*

$$\log |\Lambda| \geq -C(k,d) h'(\eta_1) \cdots h'(\eta_k) \log B, \tag{23}$$

where d is the degree of the number field generated by $\eta_1, \ldots, \eta_k$, and where

$$B = \max(|b_1|, \ldots, |b_k|, e),$$
$$h'(\eta_i) = \frac{1}{d} \max(d h(\eta_i), |\log \eta_i|, 1),$$
$$C(k,d) = 18(k+1)!\, k^{k+1} (32d)^{k+2} \log(2kd).$$

With $|\Phi| \leq \frac{1}{2}$, where

$$\Phi = e^{\Lambda} - 1 = \eta_1^{b_1} \cdots \eta_k^{b_k} - 1,$$

we have $\frac{1}{2}|\Lambda| \leq |\Phi| \leq 2|\Lambda|$ so that

$$\log \left| \eta_1^{b_1} \cdots \eta_k^{b_k} - 1 \right| \geq \log |\Lambda| - \log 2. \tag{24}$$

In view of our applications we will have $d = 2$ and $k = s + 2$ and (23) turns into

$$\log |\Lambda| \geq -C(s) h'(\eta_1) \cdots h'(\eta_k) \log B$$

with

$$C(s) = 18(s+3)!(s+2)^{s+3}(64)^{s+4} \log(4s+8).$$

Remark 1. The theorem of Baker and Wüstholz (cf. Theorem 4) [2] has a significant role in the development of linear forms in logarithms. The final structure for the lower bound for linear forms in logarithms without an explicit determination of the constant involved has been established by Wüstholz [17] and the precise determination of that constant is the central aspect of [2] (see also [3]). The reader may note that slightly sharper bounds for n_1 and z_i could be obtained by using Matveev's result [11] instead. However, the improvement is insignificant in view of our next step, that is, the use of the LLL-algorithm and the p-adic reduction method (Section 5), in which our upper bounds for n_1 and z_i are further reduced to a great extent.

We may assume that $n_3 \geq 1$ since the case $n_3 = 0$ is contained in the result due to Pink and Ziegler [13]. The first step is to deduce an upper bound for $n_1 - n_2$. For technical reasons we assume that $n_1 - n_2 \geq 5$. Let us rewrite equation (3) as

$$|\sqrt{5}\, p_1^{z_1} \cdots p_s^{z_s} \alpha^{-n_1} - 1| = \frac{|\alpha^{n_2} + \alpha^{n_3} - (\beta^{n_1} + \beta^{n_2} + \beta^{n_3})|}{\alpha^{n_1}}. \tag{25}$$

We estimate the left hand side of (25) by

$$|\sqrt{5}\, p_1^{z_1} \cdots p_s^{z_s} \alpha^{-n_1} - 1| < 5\alpha^{n_2 - n_1}.$$

Note that our assumption $n_1 - n_2 \geq 5$ was chosen such that the left hand side of equation (25) is smaller than 0.5. Using (24) we obtain from equation (25) the inequality

$$|\Lambda_1| := |\log \sqrt{5} + z_1 \log p_1 + \cdots + z_s \log p_s - n_1 \log \alpha| < 10\alpha^{n_2 - n_1}. \tag{26}$$

In order to apply Baker and Wüstholz we have to ensure that $\Lambda_1 \neq 0$. Let us assume for the moment that $\Lambda_1 = 0$. This implies that $\alpha^{n_2} + \alpha^{n_3} - (\beta^{n_1} + \beta^{n_2} + \beta^{n_3}) = 0$ which, after conjugating both sides yields $\beta^{n_2} + \beta^{n_3} - (\alpha^{n_1} + \alpha^{n_2} + \alpha^{n_3}) = 0$. Altogether we obtain $-\beta^{n_1} = \alpha^{n_1}$ which is impossible. Therefore we have $\Lambda_1 \neq 0$.

Next, we note that

$$B = \max(1, z_1, \ldots, z_s, n_1, e) < 2n_1 < n_1^2,$$
$$h'(\sqrt{5}) = \log \sqrt{5}, \qquad h'(p_i) = \log p_i, \qquad h'(\alpha) = 0.5.$$

Thus by comparing the upper bound from inequality (26) with the lower bound of $|\Lambda_1|$ from the theorem of Baker and Wüstholz in (23) we obtain the inequality

$$\log(10\alpha^{n_2 - n_1}) > -C(s) \log \sqrt{5} \log p_1 \cdots \log p_s \cdot 0.5 \log(n_1^2)$$

giving

$$n_1 - n_2 < 1.68 C(s) \log p_1 \cdots \log p_s \log n_1. \tag{27}$$

In particular, we obtain in the case that $s = 12$ the inequality $n_1 - n_2 < 3.15 \times 10^{64} \log n_1$. Let us note that the scenario $n_1 - n_2 < 5$ is also contained in the above upper bound for $n_1 - n_2$.

Next, we shall deduce an upper bound for $n_1 - n_3$. For technical reasons we assume again that $n_1 - n_3 \geq 5$. Let us rewrite equation (3) as

$$|\sqrt{5}\, p_1^{z_1} \cdots p_s^{z_s} \alpha^{-n_2} (\alpha^{n_1 - n_2} + 1)^{-1} - 1| = \frac{|\alpha^{n_3} - (\beta^{n_1} + \beta^{n_2} + \beta^{n_3})|}{\alpha^{n_1}(1 + \alpha^{n_2 - n_1})}. \tag{28}$$

We estimate the left hand side of (25) by

$$|\sqrt{5}\, p_1^{z_1} \cdots p_s^{z_s} \alpha^{-n_2} (\alpha^{n_1 - n_2} + 1)^{-1} - 1| < \frac{4\alpha^{n_3}}{\alpha^{n_1}(1 + \alpha^{n_2 - n_1})} \leq 4\alpha^{n_3 - n_1}.$$

Again our assumption $n_1 - n_3 \geq 5$ was chosen such that the left hand side of equation (28) is smaller than 0.5. Using (24) we obtain from equation (28) the inequality

$$|\Lambda_2| := \left| \log\left(\frac{\sqrt{5}}{\alpha^{n_1 - n_2} + 1}\right) + z_1 \log p_1 + \cdots + z_s \log p_s - n_2 \log \alpha \right| < 8\alpha^{n_3 - n_1}. \tag{29}$$

In order to apply Baker and Wüstholz we have to ensure that $\Lambda_2 \neq 0$. To the contrary, assume that $\Lambda_2 = 0$. This implies that $\alpha^{n_3} - (\beta^{n_1} + \beta^{n_2} + \beta^{n_3}) = 0$ which, after conjugating both sides yields $\beta^{n_3} - (\alpha^{n_1} + \alpha^{n_2} + \alpha^{n_3}) = 0$. Altogether we have

$$2 > |-(\beta^{n_1} + \beta^{n_2})| = |\alpha^{n_1} + \alpha^{n_2}| > 2$$

which is impossible. Therefore we have $\Lambda_2 \neq 0$.

We apply the theorem of Baker and Wüstholz and find again

$$B = \max(1, z_1, \ldots, z_s, n_2, e) < 2n_1 < n_1^2$$

since $z_i < 2n_1$. Besides, we have

$$\begin{aligned} h\left(\frac{\sqrt{5}}{\alpha^{n_1-n_2}+1}\right) &< h(\sqrt{5}) + (n_1 - n_2)h(\alpha) + \log 2 \\ &< \frac{\log 5}{2} + \frac{(n_1-n_2)\log\alpha}{2} + \log 2, \end{aligned}$$

so that by using (27) we have

$$\begin{aligned} h'\left(\frac{\sqrt{5}}{\alpha^{n_1-n_2}+1}\right) &= \max\left(h\left(\frac{\sqrt{5}}{\alpha^{n_1-n_2}+1}\right), \frac{1}{2}\left|\log\left(\frac{\sqrt{5}}{\alpha^{n_1-n_2}+1}\right)\right|, \frac{1}{2}\right) \\ &< \frac{\log 5}{2} + \frac{(n_1-n_2)\log\alpha}{2} + \log 2 \\ &< \log 2\sqrt{5} + \frac{1.68\log\alpha}{2} C(s)\log p_1 \cdots \log p_s \log n_1 \\ &< 0.41 C(s) \log p_1 \cdots \log p_s \log n_1. \end{aligned}$$

In the case that $s = 12$ we obtain by a similar computation the upper bound $7.68 \times 10^{63} \log n_1$. Thus by comparing the upper bound from inequality (29) with the lower bound of $|\Lambda_2|$ from the theorem of Baker and Wüstholz in (23) we obtain the inequality

$$\begin{aligned} \log(8\alpha^{n_3-n_1}) &> -C(s)h'\left(\frac{\sqrt{5}}{\alpha^{n_1-n_2}+1}\right)\log p_1 \cdots \log p_s \cdot 0.5\log(n_1)^2 \\ &> -0.41\left(C(s)\log p_1 \cdots \log p_s \log n_1\right)^2, \end{aligned}$$

and in the case that $s = 12$ we get $\log(8\alpha^{n_3-n_1}) > -1.44 \times 10^{128}(\log n_1)^2$. In the general case we get the upper bound

$$n_1 - n_3 < 0.86\left(C(s)\log p_1 \cdots \log p_s \log n_1\right)^2$$

and in the case that $s = 12$ we get $n_1 - n_3 < 3 \times 10^{128}(\log n_1)^2$.

In combination with Proposition 1 we obtain in the case that $s = 12$

$$n_1 < 4.5 \times 10^{134}(\log n_1)^4. \tag{30}$$

We note that the function $f(n) = 4.5 \times 10^{134}(\log n)^4 - n$ is decreasing for $n > 6.5 \times 10^{142}$. Solving (30) yields $n_1 < 6 \times 10^{144}$. Furthermore, Lemma 1 yields upper bounds

$$z_i < \frac{1.1n_1}{\log p_i} < \frac{6.6 \times 10^{144}}{\log p_i}$$

for all $1 \leq i \leq 12$.

Proposition 2. *If $s = 12$, then we have $n_1 < 6 \times 10^{144}$ and $z_i < \frac{6.6\times 10^{144}}{\log p_i}$ for all $1 \leq i \leq 12$.*

In the case for general s, we note that $s \geq 12$ and use the fact that $\prod_{i=1}^{12} \log p_i < 16020 < 0.003(1.5\log 12)^{12} \leq 0.003(1.5\log s)^{12}$ and $\log p_j < 1.5\log s$ for $j > 12$ from Lemma 2 *(ii)*, we have

$$\begin{aligned} n_1 &< 9500 s^3(\log n_1)^2 \times 0.86\left(C(s)\log p_1 \cdots \log p_s \log n_1\right)^2 \\ &< 9500 s^3 \times 0.86 \times (0.003)^2(1.5\log s)^{2s} C(s)^2(\log n_1)^4 \\ &< 0.1 s^3(1.5\log s)^{2s} C(s)^2(\log n_1)^4. \end{aligned}$$

Applying Lemma 3 to this inequality yields

$$n_1 < 2s^3(1.5\log s)^{2s}C(s)^2 \left(\log(26s^3(1.5\log s)^{2s}C(s)^2)\right)^4$$

which proves Theorem 1.

5. Reduction of our bounds

In this section we shall reduce the large bounds obtained by Proposition 1 in the case that $s = 12$. This will be the next step in the proof of Theorem 2. We will make use of the LLL-algorithm due to Lenstra, Lenstra and Lovász [9] to reduce our upper bounds for $n_1 - n_3$ and $z_1, \dots, z_s$. In the p-adic case we use instead of approximation lattices an idea due to Pethő and de Weger [12, Algorithm A].

5.1. **Real appoximation lattices.** Let us start by gathering some basic facts on LLL-reduced bases and approximation lattices. Let $\mathcal{L} \subseteq \mathbb{R}^k$ be a k-dimensional lattice with LLL-reduced basis $b_1, \dots, b_k$ and let B be the matrix with columns $b_1, \dots, b_k$. Moreover, we denote by $b_1^*, \dots, b_k^*$ the orthogonal basis of $\mathbb{R}^k$ which we obtain by applying the Gram-Schmidt process to the basis $b_1, \dots, b_k$. In particular, we have that

$$b_i^* = b_i - \sum_{j=1}^{i-1} \mu_{i,j} = \frac{\langle b_i, b_j \rangle}{\langle b_i^*, b_j^* \rangle}.$$

Further, we define

$$l(\mathcal{L}, y) = \begin{cases} \min_{x\in\mathcal{L}}\{\|x-y\|\}, & y \notin \mathcal{L} \\ \min_{0\neq x\in\mathcal{L}}\{\|x\|\}, & y \in \mathcal{L}, \end{cases}$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^k$. By applying the LLL-algorithm it is possible to give a lower bound for $l(\mathcal{L}, y) \geq \tilde{c}_1$ in polynomial time. (See e.g. [15, Section V.4], for details.)

Lemma 5. *Let $y \in \mathbb{R}^k$, $z = B^{-1}y$ and if $y \notin \mathcal{L}$ let i_0 be the largest index such that $z_{i_0} \neq 0$. Put $\sigma = \{z_{i_0}\}$, where $\{\cdot\}$ denotes the distance to the nearest integer, and in case that $y \in \mathcal{L}$ we put $\sigma = 1$. Moreover, let*

$$\tilde{c}_2 = \max_{1\leq j\leq k} \left\{ \frac{\|b_1\|^2}{\|b_j^*\|^2} \right\}.$$

Then we have

$$l(\mathcal{L}, y)^2 \geq \tilde{c}_2^{-1}\sigma\|b_1\|^2 = \tilde{c}_1.$$

We suppose that we are given $\eta_0, \eta_1, \dots, \eta_k \in \mathbb{R}$ and two positive constants $\tilde{c}_3, \tilde{c}_4$ such that

$$|\eta_0 + x_1\eta_1 + \cdots + x_k\eta_k| \leq \tilde{c}_3 \exp(-\tilde{c}_4 H), \tag{31}$$

where $x_i \in \mathbb{Z}$ with $1 \leq i \leq k$ are bounded by $|x_i| \leq X_i$ with X_i given upper bounds for $1 \leq i \leq k$. Set $X_0 = \max_{1\leq i\leq k}\{X_i\}$. Referring to de Weger [8] (see also [15, Section VI.3]), the basic idea is to approximate the linear form (31) by an approximation lattice. Namely, we consider the lattice $\mathcal{L}$ generated by the columns of the matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ [C\eta_1] & [C\eta_2] & \dots & [C\eta_{k-1}] & [C\eta_k] \end{pmatrix} \tag{32}$$

where $[x]$ denotes the nearest integer to x and C is a large constant usually of size about X_0^k. Let us assume that we have an LLL-reduced basis $b_1, \dots, b_k$ of $\mathcal{L}$ and that we have a lower bound $l(\mathcal{L}, y) \geq \tilde{c}_1$ with $y = (0, 0, \dots, -[C\eta_0])$. Then we have with these notations the following Lemma concerning inequality (31) (c.f. [15, Lemma VI.1]):

Lemma 6. *Assume that* $S = \sum_{i=1}^{k-1} X_i^2$ *and* $T = \frac{1+\sum_{i=1}^{k} X_i}{2}$. *If* $\tilde{c}_1^2 \geq T^2 + S$, *then we have either* $x_1 = x_2 = \cdots = x_{k-1} = 0$ *and* $x_k = -\frac{[C\eta_0]}{[C\eta_k]}$ *or*

$$H \leq \frac{1}{\tilde{c}_4}\left(\log(C\tilde{c}_3) - \log\left(\sqrt{\tilde{c}_1^2 - S} - T\right)\right).$$

We will apply Lemma 6 to inequalities (26) and (29) respectively. At first we assume that $n_1 > 4.81 \times 10^{15} > P^{10}$ so that we can apply Proposition 1 whenever necessary. We wish to deduce that $n_1 \leq 4.81 \times 10^{15}$ by the LLL-reduction method.

We recall inequality (26), that is

$$|\Lambda_1| := |\log\sqrt{5} + z_1 \log p_1 + \cdots + z_s \log p_s - n_1 \log\alpha| < 10\alpha^{n_2-n_1}$$

and assume that $s = 12$. First, we apply Lemma 6 to (26), taking $k = 13$ and $x_i = z_i$ for $i = 1, \ldots, 12$ and $x_{13} = -n_1$. For the coefficients η_i with $0 \leq i \leq 13$ we take $\eta_0 = \log\sqrt{5}$, $\eta_i = \log p_i$ for $1 \leq i \leq 12$ and $\eta_{13} = \log\alpha$. Furthermore, we have $\tilde{c}_3 = 10$ and $\tilde{c}_4 = \log\alpha$. From Proposition 2 we note that $X_0 = 9.6 \times 10^{144}$. Therefore we choose $C = 10^{1900} > X_0^{13}$. We construct the lattice generated by the columns of the matrix

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ [C\log 2] & [C\log 3] & \ldots & [C\log 37] & [C\log\alpha] \end{pmatrix} \tag{33}$$

and take $y = (0, 0, \ldots, -\left[C\log\sqrt{5}\right])$. After obtaining the LLL-reduced basis and the orthogonal basis, we refer to Lemma 5 and obtain that

$$\tilde{c}_1 = 4.84 \times 10^{291} \qquad \text{and} \qquad \tilde{c}_2 = 1.15$$

In fact the value of C is large enough such that

$$\tilde{c}_1^2 > T^2 + S.$$

Then we get by Lemma 6, either $z_1 = \cdots = z_{12} = 0$, leading to $F_{n_1} + F_{n_1} + F_{n_3} = 1$ which has only the solution $n_1 = 1$ and $n_2 = n_3 = 0$, or we have a new upper bound for $H_{1a} = n_1 - n_2$, namely

$$H_{1a} = n_1 - n_2 \leq \frac{1}{\log\alpha}\left(\log(10C) - \log\left(\sqrt{\tilde{c}_1^2 - S} - T\right)\right) < 7710. \tag{34}$$

Next, we recall inequality (29), that is

$$|\Lambda_2| := \left|\log\frac{\sqrt{5}}{\alpha^{n_1-n_2}+1} + z_1\log p_1 + \cdots + z_s \log p_s - n_2\log\alpha\right| < 8\alpha^{n_3-n_1}.$$

We apply Lemma 6 to (29), again taking $k = 13$ and $x_i = z_i$ for $i = 1, \ldots, 12$, but with $x_{13} = -n_2$. For the coefficients η_i with $0 \leq i \leq 13$ we take $\eta_0 = \log\frac{\sqrt{5}}{\alpha^{n_1-n_2}+1}$, $\eta_i = \log p_i$ for $1 \leq i \leq 12$ and $\eta_{13} = \log\alpha$. Furthermore we have $\tilde{c}_3 = 8$ and $\tilde{c}_4 = \log\alpha$. Again, Proposition 2 yields $X_0 = 9.6 \times 10^{144}$. Thus we choose $C = 10^{1900} > X_0^{13}$. We construct the lattice generated by the columns of the matrix as in (33) and take

$$y = \left(0, 0, \ldots, -\left[C\log\frac{\sqrt{5}}{\alpha^{n_1-n_2}+1}\right]\right)$$

with $0 \leq n_1 - n_2 < 7710$ from (34). We compute a LLL-reduced basis of the lattice spanned by the columns of the matrix (33) and obtain by an application of Lemma 5 to each $t = n_1 - n_2$ with $0 \leq t < 7710$ an upper bound $\tilde{c}_1$ for $l(\mathcal{L}, y)$. In all cases we get

$$\tilde{c}_2 = 1.15 \qquad \text{and} \qquad \tilde{c}_1 \geq 2.28 \times 10^{287} \qquad \text{for } 0 \leq n_1 - n_2 < 7710.$$

The value of C is large enough such that $\tilde{c}_1^2 > T^2 + S$ holds in each case.

Then we get by Lemma 6, either $z_1 = \cdots = z_s = 0$, leading to $F_{n_1} + F_{n_2} + F_{n_3} = 1$ which has only the solution $n_1 = 1$ and $n_2 = n_3 = 0$, or we have a new upper bound for $H_{1b} = n_1 - n_3$ in each case. Overall, we have

$$H_{1b} = n_1 - n_3 \leq \frac{1}{\log \alpha}\left(\log(8C) - \log\left(\sqrt{\tilde{c}_1^2 - S} - T\right)\right) < 7730 \tag{35}$$

for $0 \leq n_1 - n_2 < 7710$. The new bound for $n_1 - n_3$ in (35) immediately yields a new upper bound for n_1 in Proposition 1. We get $n_1 < 2 \times 10^{10}(\log n_1)^2$, which implies by Lemma 3 that $n_1 < 6 \times 10^{13}$ and hence $n_1 < 4.81 \times 10^{15}$.

Since the assumption under Proposition 1 that $n_1 > P^{10} = 37^{10}$ we have to deal with the case that $n_1 < 4.81 \times 10^{15} = X_{13}$ and

$$z_i < \frac{5.3 \times 10^{15}}{\log p_i} = X_i$$

for $1 \leq i \leq 12$. Now we apply the same trick once again with these new upper bounds for n_1 and z_i and obtain again a further reduction of the bounds for $n_1 - n_2$ and consequently for $n_1 - n_3$ with the procedure presented before. For this process, we take $X_0 = 7.7 \times 10^{15} = \max_{1 \leq i \leq s}\{X_i\}$ and $C = 10^{210} > X_0^{13}$. We get $\tilde{c}_1 = 2.69 \times 10^{31} > \sqrt{T^2 + S}$, $\tilde{c}_2 = 1.3$ and hence $n_1 - n_2 \leq 860$.

For each possible value for $n_1 - n_2$ with $0 \leq n_1 - n_2 \leq 860$ we compute $\tilde{c}_1$ and $\tilde{c}_2$ and obtain in each case that $\tilde{c}_2 = 1.3$ and $\tilde{c}_1 > 10^{29} > \sqrt{T^2 + S}$. Thus we conclude that $n_1 - n_3 \leq 880$ holds in each case.

Let us summarize our results so far:

Proposition 3. *We have $n_1 < 4.81 \times 10^{15}$, $n_1 - n_2 \leq 860$ and $n_1 - n_3 \leq 880$.*

5.2. **The p-adic reduction method.** In this subsection we want to apply p-adic reduction methods to inequality (12). We consider inequality (12) under the assumption that $n_1 - n_3 = t_1$ and $n_2 - n_3 = t_2$ are fixed (small) integers with $0 \leq t_2 \leq t_1 \leq 880$. The following is based on an idea due to Pethő and de Weger [12, Algorithm A]. We reproduce their idea and fit it into our framework.

Let us denote by N the upper bound for n_1 and Z the vector in which the i-th entry, Z_i, is the upper bound for z_i respectively, that is, we have $N = 4.81 \times 10^{15}$ and $Z_i = \frac{5.3 \times 10^{15}}{\log p_i}$ for $1 \leq i \leq 12$.

Let us fix an index i with $1 \leq i \leq 12$. In order to avoid an overloaded notation we drop the index i for the rest of this section. Let us consider the p-adic valuation of the left and right side of (12). We obtain

$$v_p\left(\tau(t_1, t_2)\left(\frac{\beta}{\alpha}\right)^{n_3} - 1\right) = z - v_p(\alpha^{t_1} + \alpha^{t_2} + 1) + v_p(\alpha - \beta) = z - z_0,$$

where

$$\tau(t_1, t_2) = \frac{\beta^{t_1} + \beta^{t_2} + 1}{\alpha^{t_1} + \alpha^{t_2} + 1}$$

and $z_0 := v_p(\alpha^{t_1} + \alpha^{t_2} + 1) - v_p(\alpha - \beta)$ is easily computable for fixed t_1, t_2. Let us assume that

$$z \geq \frac{3}{2} + z_0 = \tilde{c}_5 > \frac{1}{p-1} + z_0 \tag{36}$$

so that $v_p\left(\tau(t_1, t_2)\left(\frac{\beta}{\alpha}\right)^{n_3} - 1\right) = z - z_0 > \frac{1}{p-1}$. Then we may use property (8) of the p-adic logarithm to obtain

$$v_p(\log_p(\tau(t_1, t_2)) - n_3 \log_p(\alpha/\beta)) = v_p\left(\tau(t_1, t_2)\left(\frac{\beta}{\alpha}\right)^{n_3} - 1\right) = z - z_0. \tag{37}$$

Now, we distinguish two cases:

Case I: In this case we assume that $\log_p(\tau(t_1, t_2)) = 0$, which only holds when $t_1 = t_2 = 0$. In this case equation (37) turns into

$$v_p(n_3 \log_p(\alpha/\beta)) = v_p(n_3) + v_p(\log_p(\alpha/\beta)) = z - z_0$$

where $z_0 = v_p(3) - v_p(\alpha-\beta)$, that is
$$z < \frac{\log n_3}{\log p} + v_p(\log_p(\alpha/\beta)) + z_0 < \frac{\log N}{\log p} + v_p(\log_p(\alpha/\beta)) + z_0.$$
If we perform this computation for each $1 \leq i \leq 12$ we obtain a new upper bound for each z_i, denoted by Z_i in the vector Z. Thus we get
$$Z = (55, 35, 23, 20, 17, 16, 14, 14, 13, 12, 12, 12). \tag{38}$$
Then, we refer to *(iv)* of Lemma 1 and get
$$n_1 < \frac{\sum_{i=1}^{s} Z_i \log p_i}{\log \alpha} + 2.02 < 1002. \tag{39}$$

We redo all these calculations with the new-found upper bounds, that is, with $N = 1002$ and Z_i as in (38). Then we obtain in the second round
$$Z = (12, 9, 5, 5, 4, 4, 4, 4, 4, 4, 4, 3) \qquad \text{and} \qquad n_1 < 272. \tag{40}$$
Case II: Now, let us consider the case that $\log_p(\tau(t_1,t_2)) \neq 0$. Since α and β are conjugate in $\mathbb{Q}(\sqrt{5}) \subseteq \mathbb{Q}_p(\sqrt{5})$ also $\log_p \alpha$ and $\log_p \beta$ are conjugate, hence $\log_p(\alpha/\beta) = \log_p \alpha - \log_p \beta \in \sqrt{5}\mathbb{Q}_p$. Similarly we get $\log_p(\tau(t_1,t_2)) \in \sqrt{5}\mathbb{Q}_p$, since $\tau(t_1,t_2)$ is the quotient of conjugates. In particular, we get
$$\zeta = \frac{\log_p(\tau(t_1,t_2))}{\log_p(\alpha/\beta)} = \sum_{n=k}^{\infty} u_n p^n \in \mathbb{Q}_p.$$
In the case that $k < 0$, we have, from (37),
$$\begin{aligned} z - z_0 &= v_p(\log_p(\tau(t_1,t_2)) - n_3 \log_p(\alpha/\beta)) \\ &= v_p((\zeta - n_3)\log_p(\alpha/\beta)) = \overbrace{v_p(\zeta - n_3)}^{<0} + v_p(\log_p(\alpha/\beta)). \end{aligned}$$
That is we get even smaller bounds for z as we get in Case I. Therefore we may assume that $k \geq 0$ and that
$$\zeta = u_0 + u_1 p + u_2 p^2 + \cdots \in \mathbb{Z}_p$$

Let r be the smallest possible exponent such that $p^r > N$ and let $0 \leq m_0 < p^r$ be the unique integer such that $m_0 \equiv \zeta \mod p^r$. Moreover, let R be the smallest index $\geq r$ such that $u_R \neq 0$, if such an index exists. Then we get
$$\begin{aligned} z - z_0 &= v_p(\log_p(\tau(t_1,t_2))) - n_3 \log_p(\alpha/\beta)) \\ &= v_p(\zeta - n_3) + v_p(\log_p(\alpha/\beta)) \\ &\leq v_p(\zeta - m_0) + v_p(\log_p(\alpha/\beta)) \\ &= v_p(u_R p^R + \dots) + v_p(\log_p(\alpha/\beta)) \\ &= R + v_p(\log_p(\alpha/\beta)). \end{aligned}$$
Therefore we get a new upper bound for z, namely,
$$z \leq v_p(\log_p(\alpha/\beta)) + R + z_0. \tag{41}$$
Thus, we refer to (41) and get for each index $1 \leq i \leq 12$ a new upper bound z_i and therefore by Lemma 1 *(iv)* a new upper bound for n_1. We can repeat the procedure with these new, small upper bounds as long as we get smaller upper bounds for n_1.

In our situation, we perform this computation for each p_i with $1 \leq i \leq 12$ for each fixed (t_1,t_2) with $1 \leq t_1 \leq 880$ and $0 \leq t_2 \leq t_1$. We obtain for each z_i a new upper bound denoted by Z_i in the vector Z and get
$$Z = (73, 46, 31, 27, 22, 21, 18, 18, 17, 12, 16, 17). \tag{42}$$
By *(iv)* of Lemma 1 we get
$$n_1 < \frac{\sum_{i=1}^{12} z_i \log p_i}{\log \alpha} + 2.02 < 1300.$$

We can further reduce these new upper bounds for z_i and n_1, by performing the same procedure in Case II once again, taking $N = 1300$ and Z_i as in (42). In the second round, we obtain

$$Z = (30, 22, 13, 11, 9, 9, 9, 8, 8, 7, 8, 7) \quad \text{and} \quad n_1 < 590. \tag{43}$$

This bound is small enough to perform a computer enumeration.

Next, let us discuss the case in which R does not exist. In this case we would obtain that $\zeta = m_0$ is an integer, hence

$$\log_p(\tau(t_1, t_2)) = m_0 \log_p(\alpha/\beta),$$

and (37) turns into

$$\begin{aligned} z - z_0 &= v_p(\log_p(\tau(t_1, t_2)) - n_3 \log_p(\alpha/\beta)) \\ &= v_p(\zeta - n_3) + v_p(\log_p(\alpha/\beta)) \\ &= v_p(m_0 - n_3) + v_p(\log_p(\alpha/\beta)), \end{aligned}$$

that is

$$\begin{aligned} z &< \frac{\log |m_0 - n_3|}{\log p} + v_p(\log_p(\alpha/\beta)) + z_0 < \frac{\log p^r}{\log p} + v_p(\log_p(\alpha/\beta)) + z_0 \\ &= r + v_p(\log_p(\alpha/\beta)) + z_0. \end{aligned}$$

We perform this computation for each $1 \le i \le 12$ with $1 \le t_1 \le 880$ and $0 \le t_2 \le t_1$ to obtain a new upper bound for each z_i, denoted by Z_i in the vector Z. Thus we get

$$Z = (64, 39, 31, 23, 22, 18, 16, 18, 15, 15, 16, 13). \tag{44}$$

Then, we refer to *(iv)* of Lemma 1 and get

$$n_1 < \frac{\sum_{i=1}^{s} Z_i \log p_i}{\log \alpha} + 2.02 < 1220.$$

We redo all these calculations with the new found upper bounds, that is, with $N = 1220$ and Z_i as in (44). Then we obtain in the second round

$$Z = (22, 13, 13, 8, 9, 6, 6, 8, 6, 7, 8, 4).$$

By *(iv)* of Lemma 1 we get $n_1 < 480$ which is smaller than the upper bound in (43).

Finally, let us consider, on the contrary to (36), that $z < \tilde{c}_5 = \frac{3}{2} + z_0 = \frac{3}{2} + v_p(\alpha^{t_1} + \alpha^{t_2} + 1) - v_p(\alpha - \beta)$ for $0 \le t_2 \le t_1 \le 880$. We obtain

$$Z = (11, 7, 9, 5, 7, 4, 4, 6, 4, 5, 6, 3).$$

By *(iv)* of Lemma 1 we get $n_1 < 330$ which is smaller than the upper bound in (43). Therefore we have proved that in any case all solutions to (3) satisfy $n_1 < 590$.

6. Computer enumeration

By computer enumeration, we completely solve Diophantine equation (3) for $s = 12$, taking into the account that $n_1 < 590$. We proved Theorem 2.

We refrain from giving a list of all the 855 solutions to (3). Note that, a brute force computer search for all solutions to (3) with $0 \le n_3 \le n_2 \le n_1 < 590$ takes a few minutes on a usual PC. However, looking through the solutions to (3) we noticed the following facts:

- $1 \le n_1 \le 48$, $0 \le n_2 \le 34$ and $0 \le n_3 \le 24$
- $0 \le z_1 \le 14$
- $0 \le z_2 \le 6$
- $0 \le z_3, z_4 \le 4$
- $0 \le z_5, z_6, z_7, z_8, z_9 \le 3$
- $0 \le z_{10}, z_{12} \le 2$
- $0 \le z_{11} \le 1$

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